The computation of transfer maps, Evens norm maps and Steenrod operations

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Gutachter:

1. Prof. Dr. David J. Green, Friedrich-Schiller-Universität Jena
2. Prof. Dr. Jon F. Carlson, Universität Georgia, Athens

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Lebenslauf

Vorname/Name: Thanh Tung Vo

Geburtsdatum: 14.03.1977

Geburtsort: Hue, Vietnam

Staatsangehörigkeit: Vietnamese

1983-1995: Grundschule und Oberschule

1995-1999: Studium der Mathematik, Universität Hue, Vietnam

2005: Mathematik Magister, Universität Hue, Vietnam

2000-2007: Wissenschaftlicher Mitarbeiter, Universität Hue, Vietnam

2008-2011: Promotionsstudent, Friedrich-Schiller-Universität Jena, Deutschland

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Thanh Tung Vo

Abstract

The mod-p cohomology of any group is a module over the Steenrod algebra, hence it is naturally that this algebra plays a great important role in the theory of group cohomology. This dissertation describes the method and its implementation in the computer algebra system Sage for computing the Steenrod operations on the mod-2 cohomology rings of finite groups.

We often use the detection method to relate the Steenrod operations on the cohomology rings of a given group to the ones of its subgroups. In case the detection method does not work, we shall try the computations using combinatorially the inflation, the transfer map and the comodule map. There are some certain cases that the above methods are not enough but the order of group is appropriate, then we shall use the Evens norm map to generate the Steenrod operations. In the dissertation, we also describe our implementations to calculate the transfer and the Evens norm map.

We obtain from the first run of our Sage package the Steenrod operations for many groups. Especially, we have determined the Steenrod operations on the mod-2 cohomology ring of the Mathieu group M_{22}, and 49 groups of the 51 groups of order 32. The remaining cases of the groups of order 32 will be completed soon.

Zusammenfassung

Die p-modulare Kohomologie einer Gruppe ist ein Modul über der Steenrod-Algebra, dass sie eine maßgebliche Rolle in der Gruppenkohomologietheorie spielt ist daher selbstverständlich. Diese Dissertation beschreibt die Methode und ihre Computerimplementierungen in Sage zur Berechnung der Steenrod-Operationen in den mod-2 Kohomologieringen endlicher Gruppen.

Oft verwenden wir das Nachweisverfahren um die Steenrod-Operationen der Kohomologieringe einer bestimmten Gruppe zu den Operationen ihrer Untergruppen in Beziehung zu setzen. Im Falle dass die Nachweismethode nicht funktioniert, versuchen wir die Berechnungen mit Hilfe der Inflation, der Transfer- und der Komodul-Abbildung. Es gibt einige bestimmte Fälle, in denen die oben genannten Methoden nicht ausreichen aber die Ordnung der Gruppe geeignet ist, die Evens-Norm-Abbildung zu verwenden, um die Steenrod-Operationen zu erzeugen. In dieser Dissertation, beschreiben wir auch unsere Implementierungen um die Transfer- und die Evens-Norm-Abbildung zu berechnen.

Wir erhalten von der ersten Auflage unseres Sage-Paketes die Steenrod- Operationen für viele Gruppen. Insbesondere haben wir die Steenrod-Operationen für die mod-2 Kohomologieringe der Mathieu-Gruppe M_{22}, und der 49 Gruppen von den 51 Gruppen der Ordnung 32 bestimmt. Die zwei Übrigen der Gruppen der Ordnung 32 werden bald fertig gerechnet.

To Dieu My, Topo and Topy.

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Finally, I dedicate this dissertation to my family, especially to my wife Dieu My, my son Topo and my daughter Topy for their patience, encouragement and unconditional love.

Contents

Introduction

This dissertation is devoted to the computation of the Steenrod operations on the mod-2 cohomology rings of finite groups and the implementations of the transfer and the norm map. The motivation for our computational work is as follows.

Initially, we started our research by considering Carlson's depth-conjecture, which is stated in [7] that if G is finite group such that the cohomology ring $\mathrm{H}^*(G, \mathbb{F}_p)$ is not Cohen-Macauley, then the depth d of $\mathrm{H}^*(G, \mathbb{F}_p)$ is the largest number that $\mathrm{H}^*(G, \mathbb{F}_p)$ is detected on the centralizers of the elementary abelian p-subgroups of G of rank d. In [16], Green proved a weak form of the conjecture in the case G is a p-group and the depth $\mathrm{H}^*(G, \mathbb{F}_p)$ equals the rank of the center of G. Then Kuhn [20] gave an improvement of the result of Green to arbitrary group G. We tried to check the conjecture by doing the calculations on some non-prime power groups and this work led us to the Mathieu group M_{22}, an important sporadic simple group. The mod-2 cohomology ring of this group, up to nilpotence, has been determined by Adem and Milgram in [2]. Nevertheless, the explicit structure of the cohomology ring of M_{22} is incomplete and needs to be implemented.

We know that determining the explicit structure of the cohomology ring of a finite group, in most cases, is a hard work. Carlson first represented in [8] a feasible and systematical computer calculation method of group cohomology by using minimal projective resolution approach. In [15], Green improved Carlson's method by developing two new kinds of Gröbner bases for modules over p-groups and for graded commutative ring, his implementation was written in the C programming language, using the library of the package "C-MeatAxe". Later, S. King wrapped the C-routines by Green into the computer algebra system Sage and added some new features. Green and King have applied some improvements of Carson's completeness criterion to compute the modular cohomology rings for all 2-groups of order 128 and for several larger groups (see [18]).

Back to the point of the Mathieu group M_{22}, we are interested in the ring properties of the cohomology of this group. The cohomology structure of a finite group is both as a ring with the cup product and as a module over the Steenrod algebra. As in the paper of Adem and Milgram [2], the Steenrod operations are helpful to determine the generators of the cohomology ring of M_{22}. However, calculating Steenrod operations is quite a complicated work even when we have known the cohomology ring already. Using publicly available essential ideal computations by Green (see [14]), we showed that the cohomology rings of M_{22} and of its Sylow 2-subgroup are detected on the subgroups of order 16. And these groups are small enough to be treated explicitly. Moreover, Green and King successfully determined, by using the stable elements method, the cohomology rings of some non-prime power groups, especially of the groups M_{22} and M_{23}. Hence, we demonstrated that it is feasible to determine the Steenrod algebra structure for M_{22} on computer.

We choose the computer algebra system Sage to implement our method for calculating Steenrod operations. One reason for this choice is we want to use the cohomology rings of finite groups from the Sage package of Green and King [18]. Another reason is there is a package of Palmieri, which is available in Sage, dealing well with the mod-p Steenrod algebra. Let G be a finite group. By the Cartan formula, we will compute the action of the basic Steenrod squares Sq^{2^i} on the generators of the cohomology ring $\mathrm{H}^*(G, \mathbb{F}_2)$. If G is detected on a family of proper subgroups, then Steenrod squares on $\mathrm{H}^*(G, \mathbb{F}_2)$ are determined from the ones on the cohomology rings of its subgroups by using the detection method. On the contrary, we will use combinatively the restriction, inflation, the transfer or the comodule map to compute for certain cases. In the case G is a 2-group of order not bigger than 32, we may use the Evens norm map to calculate all basic actions instead of the method above. In this dissertation, we also discuss the methods to calculate the transfer and the Evens norm map and describe our implementations of these maps.

Before us, Steenrod operations on the cohomology rings of 2-groups of small orders were calculated by Rusin [25], and by Guillot [19], but not completely. In our calculations, we have gone further by calculating for all 2-groups of order less than or equal 32, except the two cases in order 32 which will hopefully be soon updated, more than 210 groups of the 267 2-groups of order 64 and several 2-groups of order 128. Especially, the Steenrod squares on the cohomology rings of several non-prime power groups such as Mathieu groups were determined.

The organization of the dissertation is as follows. In Chapter 1, we give an overview of group cohomology and then discuss the computation of the cohomology rings of finite groups. In Chapter 2, we shall introduce our implementation on the calculation of the transfer map. We discuss the computation of the Evens norm map in Chapter 3. We devote Chapter 4 to the strategies for computing Steenrod operations on the mod-2 cohomology rings of finite groups. Finally, we shall describe in Chapter 5 some selective results on Steenrod operations that we archived to date from the first run of our package.

Throughout the dissertation, p is a prime number, k is a commutative ring or a field of characteristic p, and G is a finite group, unless otherwise specified. The notation $SmallGroup(n, r)$ denotes a group of order n and number r in the small group library of the computer algebra system GAP.

Chapter 1

Group cohomology and the computation

In this chapter we give a brief introduction to group cohomology and then discuss the computation of the cohomology rings of groups. In the first section, an overview of group cohomology is given. Most of the definitions and results in this part are well-known, and may be found in Benson [4], Carlson et al. [10] and Evens [13]. In the remaining part, we shall present a brief summary of the computer calculation of the cohomology of finite groups, which uses the minimal resolutions approach.

1.1 Overview of group cohomology

Let G be a finite group, and let k be a commutative ring. Suppose that A is a $\mathrm{k}\,(G)$-module. The group cohomology of G with coefficients in A is simply defined to be

$$\mathrm{H}^*(G, A) = \mathrm{Ext}^*_{\mathrm{k}(G)}(k, A).$$

From homological algebra, the above structure is defined as follows. Choose a $\mathrm{k}\,(G)$-projective resolution $\epsilon : P_* \to k$ of the trivial $\mathrm{k}\,(G)$-module k with augmentation

$\epsilon : P_0 \to k$. Then $\mathrm{H}^*(G, A)$ is the cohomology of the cochain complex $\mathrm{Hom}_{\mathrm{k}(G)}(P_*, A)$.

Notice that the construction of $\mathrm{H}^*(G, A)$ does not depend on the ring k and the choice of the $\mathrm{k}\,(G)$-projective resolution P_*. Therefore, we may sometime calculate the cohomology of a group efficiently by choosing an appropriate resolution. To calculate group cohomology in low dimension, we usually choose the *bar resolution* , which is defined as follows. For each $n \geq 0$, let B_n denote the free $\mathrm{k}\,(G)$-module with basis G^n. When $n = 0$, the basis of B_0 has only one element denoted by $[.]$. Define the augmentation $\epsilon : B_0 \longrightarrow k$ by $\epsilon([.]) = 1$.

Definition 1.1.1. The bar resolution of G is the free resolution B_* with the augmentation ϵ and with the differentials defined by

$$\begin{aligned}\partial_n(g_1, g_2 \dots, g_n) = g_1(g_2, \dots, g_n) \\ &+ \sum_{i=1}^{n-1}(-1)^i(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_n) \\ &+ (-1)^n(g_1, g_2, \dots, g_{n-1}).\end{aligned}$$

In some specific cases, for example computer calculations, the minimal resolution is a clever choice for the computations of group cohomology however. We will discuss this subject in the next section in detail.

We will now define products in group cohomology using the bar resolution. Let G be a group, let A and B be $\mathrm{k}\,(G)$-modules. Let $B_* \to k$ be the bar resolution of G. Define homomorphisms

$$\begin{aligned}\times : \mathrm{H}^r(G, A) \otimes_k \mathrm{H}^s(G, B) &\longrightarrow \mathrm{H}^{r+s}(G, A \otimes_k B) \\ [u] \otimes [v] &\longmapsto [u] \times [v] := [u \times v]\end{aligned}$$

where $u \times v$ given by

$$(u \times v)(g_1, g_2, \dots, g_{r+s}) = u(g_1, \dots, g_r) \otimes (g_1 \cdots g_r)v(g_{r+1}, \dots, g_{r+s}),$$

for all $(g_1, g_2, \ldots, g_{r+s}) \in B_{r+s}$. The collection of homomorphisms above is called *cup product*. Suppose that $\theta : A \otimes_k B \longrightarrow C$ is a $\mathrm{k}\,(G)$-homomorphism such that $\theta(g(a \otimes b)) = \theta(ga \otimes gb) = g\theta(a \otimes b)$, for all $g \in G, a \in A, b \in B$. Then by functorality there is a induced map $\theta^* : \mathrm{H}^*(G, A \otimes_k B) \longrightarrow \mathrm{H}^*(G, C)$. For all $\alpha \in \mathrm{H}^*(G, A), \beta \in \mathrm{H}^*(G, B)$, the product

$$\alpha\beta := \theta^*(\alpha \times \beta)$$

is called θ-cup product in group cohomology. The map θ is called a G-pairing. In particular, if $A = B = C$, then A is called a kG-ring with the multiplication map θ, and $\mathrm{H}^*(G, A)$ with the cup product is an associative graded ring.

Proposition 1.1.2. *If A is a commutative $\mathrm{k}\,(G)$-ring, then $\mathrm{H}^*(G, A)$ is a commutative graded ring , i.e. $\alpha\beta = (-1)^{rs}\beta\alpha$, for all $\alpha, \beta \in \mathrm{H}^*(G, A)$.*

Proof. It is straightforward from the properties of the cup product. See Evens [13], Section 3.1 for the proof of properties of the cup product. □

Moreover, when the ring of coefficients A identifies with a noetherian base ring k, then we have

Theorem 1.1.3 (Evens-Venkov)**.** *If k is a noetherian ring on which G acts trivially, then $\mathrm{H}^*(G, k)$ is a finitely generated k-algebra. Moreover, if M is a finitely generated $\mathrm{k}\,(G)$-module, then $\mathrm{H}^*(G, M)$ is a finitely generated $\mathrm{H}^*(G, k)$-module.*

Proof. See the proof of Theorem 6.5.1 in Carlson et al. [10]. □

Let $M = k_H^{\uparrow G}$ be the induction of the trivial $\mathrm{k}\,(H)$-module k. By the Eckmann-Shapiro Lemma (see e.g. [3], p. 47), $\mathrm{H}^*(G, k_H^{\uparrow G}) \cong \mathrm{H}^*(H, k)$. Then we have the following.

Corrolary 1.1.4. *Suppose that $H \leq G$ and that k is a noetherian ring. Then $\mathrm{H}^*(H,k)$ is a finitely generated $\mathrm{H}^*(G,k)$-module via the restriction map.*

Note that the most interesting cases of commutative $\mathrm{k}\,(G)$-rings k are $k = \mathbb{Z}$ or $\mathbb{F}_p$, where we consider those rings as trivial $\mathrm{k}\,(G)$-modules.

By definition, it is clear that $\mathrm{H}^*(-,-)$ is a bifunctor, covariant in the first variable and contravariant in the second, from the category of pairs group-module to the category of graded abelian groups. We can describe this functor as follows. Let $\phi : G' \longrightarrow G$ be a group homomorphism. Let A be a $\mathrm{k}\,(G)$-module and A' a $\mathrm{k}\,(G')$-module. Through the homomorphism ϕ, we may consider A as a $\mathrm{k}\,(G')$-module. Suppose that $f : A \longrightarrow A'$ is a $\mathrm{k}\,(G')$-module homomorphism, i.e. $f(\phi(x')a) = x'f(a)$ for $x' \in G'$ and $a \in A$. Then the pair of homomorphisms (ϕ, f) induces a group homomorphism

$$\mathrm{H}^*(\phi, f) : \mathrm{H}^*(G, A) \longrightarrow \mathrm{H}^*(G', A').$$

We define now two basic maps on group cohomology induced from group homomorphisms by cohomology functors. Let G be a group and H a subgroup of G. Suppose that A is a $\mathrm{k}\,(G)$-module, then it may be viewed as a $\mathrm{k}\,(H)$-module through the inclusion $i : H \longrightarrow G$. By functorality, there is an induced map

$$\mathrm{H}^*(i, id_A) : \mathrm{H}^*(G, A) \longrightarrow \mathrm{H}^*(H, A)$$

which is called *restriction* and denoted res^G_H. Suppose now that H is a normal subgroup of G. Let A be a $\mathrm{k}\,(G/H)$-module, then A may be considered as a $\mathrm{k}\,(G)$-module. Let $\pi : G \longrightarrow G/H$ be the canonical projection of G onto G/H. Similarly, there is an induced map

$$\mathrm{H}^*(\pi, id_A) : \mathrm{H}^*(G/H, A) \longrightarrow \mathrm{H}^*(G, A)$$

which is called *inflation* and denoted $\inf_{G/H}^{G}$.

1.2 Minimal resolutions

In this section, G is a finite group and k is a field. We refer to Carlson et al. [10], Chapter 3, Section 2 for the definitions and properties throughout this section.

Definition 1.2.1. Let M be a finitely generated $\mathrm{k}\,(G)$-module. A *minimal projective resolution* of M is a projective resolution

$$(P_*, \varepsilon): \qquad \ldots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \ldots \to P_1 \to P_0 \xrightarrow{\varepsilon} M \to 0$$

of M such that $\partial_n(P_n) \subset \mathrm{Rad}(P_{n-1})$ for all $n > 0$.

The existence of the minimal projective resolutions is asserted by the following lemma.

Lemma 1.2.2. *Let M be a finitely generated $\mathrm{k}\,(G)$-module. Then M has a minimal projective resolution.*

The statements in (ii) and (iii) of the following proposition show the advantage of minimal resolutions in computing group cohomology.

Proposition 1.2.3. *Let M be a finitely generated $\mathrm{k}\,(G)$-module, and (P_*, ε) a projective resolution of M. Then the following statements are equivalent.*

(i). (P_*, ε) is minimal projective resolution of M.

(ii). If S is a simple $\mathrm{k}\,(G)$-module, then for all $n > 0$

$$\mathrm{Ext}^n_{\mathrm{k}(G)}(M, S) = \mathrm{Hom}_{\mathrm{k}(G)}(P_n, S)$$

(iii). If S is a simple $\mathrm{k}\,(G)$*-module, then for all* $n \geq 0$*, the map*

$$\partial^* : \mathrm{Hom}_{\mathrm{k}(G)}(P_n, S) \longrightarrow \mathrm{Hom}_{\mathrm{k}(G)}(P_{n+1}, S)$$

is the zero map.

(iv). Let (P'_*, ε') *be another projective resolution of* M*. Then the chain map* μ : $(P'_*, \varepsilon') \to (P_*, \varepsilon)$ *that lifts the identity map on* M *is surjective.*

(v). Let (P'_*, ε') *be another projective resolution of* M*. Then the chain map* ν : $(P_*, \varepsilon) \to (P'_*, \varepsilon')$ *that lifts the identity map on* M *is injective.*

1.3 A brief summary of the computation

Roughly speaking, the computation of group cohomology using minimal resolutions approach is an approximate method to compute the mod-p cohomology ring of p-groups from a suitably large initial segment of the minimal projective resolution. The method was first described by Carlson in [8] as a feasible and systematical computer calculation method of group cohomology. Later, Green [15] developed the method by representing the two new kinds of Gröbner bases for modules over p-groups and for graded commutative ring.

The method of Carlson is described explicitly as follows. Suppose that G is a finite p-group and k is field of characteristic p. Then by Theorem 1.1.3, the cohomology ring $\mathrm{H}^*(G, k)$ is a finitely generated k-algebra. So we reasonably want to represent $\mathrm{H}^*(G, k)$ as a polynomial ring over k in a finite set of generators (actually it is a Θ-algebra, see e.g. [15]) and modulo a set of relations. First of all we need to build a projective resolution for our computation. From the advantages of minimal resolution mentioned in Section 1.2, we choose to construct the minimal projective resolution of the trivial $\mathrm{k}\,(G)$-module k.

Constructing minimal resolutions.

The construction of the minimal projective resolution of the trivial $\mathrm{k}\,(G)$-module k is somehow easy to see. Set $P_0 = \mathrm{k}\,(G)$ and consider it as a $\mathrm{k}\,(G)$-module of rank 1. Then the augmentation $\varepsilon : P_0 \to k$ is also a projective cover of k. We compute the kernel of ε and denote it by $\Omega(k)$. Note that if S is a generating set for the group G, then $S - 1 = \{s - 1 \mid s \in S\}$ is a generating set for $\Omega(k)$. Suppose that there are rk_1 elements $s_1, \ldots, s_{rk_1} \in S$ such that $s_1 - 1, \ldots, s_{rk_1} - 1$ form a minimal generating set for $\Omega(k)$ as a $\mathrm{k}\,(G)$-module. Let $P_1 = \bigoplus_{j=1}^{rk_1} \mathrm{k}\,(G) u_j$ be a free $\mathrm{k}\,(G)$-module with basis $u_1, \ldots, u_{rk_1}$. Then we have a projective cover of $\Omega(k)$

$$d_1 : P_1 \longrightarrow \Omega(k)$$

given by $d_1(u_j) = s_j - 1$ for $j = 1, \ldots, rk_1$. Generally, suppose that $\Omega^n(k)$ is the kernel of the projective cover $d_n : P_{n-1} \to \Omega^{n-1}(k)$ and $\Omega^n(k)$ has a minimal generating set $v_1, \ldots, v_{rk_n}$. Then we have a projective cover $P_n = \bigoplus_{j=1}^{rk_n} \mathrm{k}\,(G) w_j$ of $\Omega^n(k)$

$$d_n : P_n \longrightarrow \Omega^n(k)$$

given by $d_1(w_j) = v_j$ for $j = 1, \ldots, rk_n$. We repeat this process N times then finally have a portion

$$P_N \xrightarrow{\partial_N} P_{N-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \to k \to 0 \tag{1.3.1}$$

of the minimal projective resolution of k.

Note that to construct the minimal resolution we determine the minimal generating set for the kernel of a $\mathrm{k}\,(G)$-linear map between free $\mathrm{k}\,(G)$-modules iteratively.

Carlson et al. in [9] use the linear algebra method to do that. However for large groups this method becomes less efficient, because this method works over k-vector spaces with big dimensions. Green in [15] developed an efficient method based on a new kind of Gröbner bases for module. This method allows us to work over $\mathrm{k}(G)$-modules, hence for large groups the program saves a considerable amount of space. In Chapter 4, we will take advantage of this method in constructing the tensor product of minimal resolutions.

Calculating cup products

Suppose that we could construct a minimal resolution of the $\mathrm{k}(G)$-module k as above out to the Nth term. It follows from Proposition 1.2.3 that

$$\mathrm{H}^n(G,k) \cong \mathrm{Ext}^n_{\mathrm{k}(G)}(k,k) \cong \mathrm{Hom}_{\mathrm{k}(G)}(P_n,k)$$

for all n. We will use the following method, which is described in Carlson et al. [9], to determine cup products in $\mathrm{H}^*(G,k)$. For a cohomology element $\phi \in \mathrm{H}^n(G,k)$, we may consider ϕ to be the cocycle $\phi : P_n \to k$. The cocycle ϕ can be always lifted to a chain map as in the diagram below

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+m} \arrow[r, "\partial_{n+m}"] \arrow[d, "\phi_m"] & P_{n+m-1} \arrow[r] \arrow[d, "\phi_{m-1}"] & \cdots \arrow[r] & P_n \arrow[r] \arrow[d, "\phi_0"] \arrow[dr, "\phi"] & \cdots \arrow[r] & P_0 \arrow[r] & k \arrow[r] & 0 \\
\cdots \arrow[r] & P_m \arrow[r, "\partial_m"] & P_{m-1} \arrow[r] & \cdots \arrow[r] & P_0 \arrow[r] & k. & & &
\end{tikzcd}$$

Now for any cocycle $\psi : P_m \to k$ as a cohomology class of degree m, the cup product $\psi\phi$ is the composition $\psi \circ \phi_m : P_{n+m} \to k$.

The main work in this stage is to solve the lifting problem. Suppose P_* and Q_* are the chain complexes of projective $\mathrm{k}(G)$-modules and $\phi_{n-1} : P_{n-1} \to Q_{n-1}$ is a

$\mathrm{k}\,(G)$-linear map such that $\mathrm{Im}(\phi_{n-1} \circ \partial_n) \subseteq \mathrm{Im}(\partial'_n)$. A lift of ϕ_{n-1} to degree n is a $\mathrm{k}\,(G)$-linear map $\phi_n : P_n \to Q_n$ which makes the diagram

$$\begin{tikzcd} P_n \arrow[r, "\partial_n"] \arrow[d, "\phi_n"] & P_{n-1} \arrow[d, "\phi_{n-1}"] \\ Q_n \arrow[r, "\partial'_n"] & Q_{n-1} \end{tikzcd}$$

commute. For each element $u \in P_n$, the lift ϕ_n is determined by setting

$$\phi_n(u) := \text{ a preimage of } \phi_{n-1}(\partial_n(u)) \text{ under } \partial'_n.$$

Again the Gröbner basis method is the best choice for computing preimages. This works especially well for high degrees.

Representing the cohomology ring out to a given degree

For a given N, suppose that we have constructed the minimal resolution of k out to the Nth term as in (1.3.1). Now we have a set of generators for the cohomology ring consisting of elements of degrees at most N. There is a method to reduce the set of generators and determine a set of relations, which is described in Carlson et al. [9]. Green in [15] developed a new kind of Gröbner bases for graded-commutative algebras to calculate a minimal generating set and a minimal set of relations. At this point we have a representation of the cohomology ring by the generating set and the relations out to degree N. We need a criterion to decide if these generators and relations are enough to represent the whole cohomology ring. Carlson is the first author who gave a criterion to test for the completion of the calculation (see [8]). Later, there are some other improvements for the criterion of Carlson, as we can see in the next section. Green and King in [18] have made use of these improvement criteria to compute the modular cohomology rings for all 2-groups of order 128 and for several larger groups.

1.4 The completeness criteria

In this part we will give a very brief introduction to the criteria that Benson [5] had proposed, and then Green and King [17] improved and applied to calculate the cohomology rings of all groups of order 128. For the transparency of the theorems below, we introduce here the necessary concepts which are defined in Benson [5] and recalled in [17].

Definition 1.4.1. Let k be a field of characteristic $p > 0$. Let $A = \oplus_{n \geq 0} A_n$ be a connected Noetherian graded commutative k-algebra. For $a \in A_n$, we write $|a| = n$. Denote by A_+ the maximal ideal of A consisting of the elements of positive degree.

(i). A sequence of elements $h_1, \ldots, h_r \in A_+$ is called filter-regular if for each $1 \leq i \leq r$ the annihilator of h_i in the quotient ring $A/(h_1, \ldots, h_{i-1})$ has bounded degree.

(ii). A system of parameters $h_1, \ldots, h_r$ which is filter-regular is said to have type $(d_0, d_1, \ldots, d_r) \in \mathbb{Z}^{r+1}$ if

$$d_0 \geq -1, d_{i-1} - 1 \leq d_i \leq d_{i-1} \text{ for each } 1 \leq i \leq r,$$

and the annihilator of h_{i+1} in $A/(h_1, \ldots, h_i)$ lies in degrees $\leq d_i + \sum_{j=0}^{i} |h_i|$ for all $0 \leq i \leq r$, where $h_{r+1} = 0$.

Definition 1.4.2. Let$M$ be a finitely generated graded H-module, we write $H_m^{i,n} M$ for the degree n part of the i^{th} local cohomology module of M. Let H be a graded ring satisfying Condition 2.2 in Benson [5]. We set

$$a_m^i(M) = \max\{n \in \mathbb{Z} \mid H_m^{i,n} M \neq 0\}.$$

Let G be a finite group. Let $P = k[x_1, \dots, x_n]$ be a graded commutative polynomial ring. Suppose $\deg x_i \le \deg x_j$ if $i < j$. We want to represent the cohomology ring $\mathrm{H}^*(G, k)$ in the form $\mathrm{H}^*(G, k) = P/I$, with I is an ideal in P. Suppose that we have computed to degree N of the cohomology ring $\mathrm{H}^*(G, k)$ in the sense that we have an approximation $R_N = Q/J$, where Q and J satisfy the following.

(i). $Q = k[x_1, \dots, x_m] \subseteq P$ with $\deg x_j \le N$.

(ii). J is the ideal of Q generated by all homogeneous elements of I of degree at most N.

As in [5], we write $\tau_N \mathrm{H}^*(G, k) = R_N$. Then there is a unique homomorphism $\tau_N \mathrm{H}^*(G, k) \to \mathrm{H}^*(G, k)$ given by mapping $x_i + J$ to x_i. The theorem below gives a criterion for testing when the map is an isomorphism.

Theorem 1.4.3 (Benson [5]). *Let $r = r_p(G) > 1$. Suppose that $\zeta_1, \dots, \zeta_r \in \tau_N \mathrm{H}^*(G, k)$ form a filter-regular homogeneous set of parameters with $|\zeta_i| = n_i \ge 2$. Suppose further that their images in $\mathrm{H}^*(G, k)$ form a homogeneous system of parameters. Set*

$$\alpha = \max_{0 \le i \le r-2} \{a^i_m(\tau_N \mathrm{H}^*(G, k)) + i\} \tag{1.4.1}$$

($\alpha = \infty$ if $\tau_N \mathrm{H}^*(G, k)$ has depth at least $r - 1$). If

$$N > \max\{\alpha, 0\} + \sum_{j=1}^{r} n_j - 1 \tag{1.4.2}$$

then the map $\tau_N \mathrm{H}^*(G, k) \to \mathrm{H}^*(G, k)$ is an isomorphism.

A modification of Benson's criterion by Green and King allows us to reduce the test for completion to much smaller N.

Theorem 1.4.4 (Green-King [17]). *Let G be a finite group, k a field of characteristic p, and K/k a field extension. Suppose that $r = r_p(G) > 2$. Suppose that $\zeta_1, \ldots, \zeta_r \in \tau_N \operatorname{H}^*(G,k)$ and $\kappa_1, \ldots, \kappa_r \in \tau_N \operatorname{H}^*(G,K)$ have the following properties:*

(i). Each system is a filter-regular homogeneous set of parameters in its respective ring. Let $(d_0, \ldots, d_r)$ be the type of $\zeta_1, \ldots, \zeta_r$.

(ii). The images of the ζ_i form a homogeneous set of parameters in $\operatorname{H}^*(G,k)$.

(iii). Set $n_i = |\kappa_i|$ and $\alpha' = \max_{0 \geq i \geq n-2} \{d_i + i\}$. Then $n_i \geq 2$ for all i, and

$$N > \max\{\alpha', 0\} + \sum_{j=1}^{n} n_j - 1. \tag{1.4.3}$$

Then the map $\tau_N \operatorname{H}^*(G,k) \to \operatorname{H}^*(G,k)$ is an isomorphism.

Moreover, if a Sylow p-subgroup of G has centre of p-rank at least two, then in Eq. (1.4.3) we only have to require $\geq$.

Chapter 2

The transfer map

In this chapter, we introduce our implementation on the calculation of the transfer map. The computer calculation of the transfer map was presented by M. Zhang in [27] (or briefly described in Carlson et al. ([10], p. 322-323)) as a feature in a computer project of Carlson, which computes the mod-2 cohomology rings of all 2-groups of order 64. In this chapter, we shall introduce our implementation of this feature to the computer project of D. Green and S. King [18]. And in Chapter 4, we shall make use of this feature to the calculation of Steenrod operations.

In Section 2.1 we will provide the definition of the transfer map and some important properties. For the materials of the transfer map, we refer the reader to Adem and Milgram [1], Carlson et al. [10] and Evens [11]. In Section 2.2, we will describe our implementation of transfer maps in mod-p cohomology rings of p-groups.

2.1 Definition and properties

Let G be a group, H a subgroup of finite index, and A a $\mathrm{k}(G)$-module. Let T be a set of left coset representatives of H in G. Suppose that $P_* \to k$ is a $\mathrm{k}(G)$-projective resolution, then it is also a $\mathrm{k}(H)$-projective resolution. Define a map

$$\mathrm{Tr} : \mathrm{Hom}_{\mathrm{k}(H)}(P_*, A) \longrightarrow \mathrm{Hom}_{\mathrm{k}(G)}(P_*, A) \tag{2.1.1}$$

by

$$\mathrm{Tr}(f)(x) = \sum_{t\in T} tf(t^{-1}x),$$

for $f \in \mathrm{Hom}_{\mathrm{k}(H)}(P_*, A)$.

It is not hard to check that the map is well-defined, independent of the choice of the resolution or the coset representatives, and it commutes with the differentials in the complexes in two sides. So it induces a homomorphism

$$\mathrm{tr}_H^G : \mathrm{H}^*(H, A) \longrightarrow \mathrm{H}^*(G, A)$$

which is called *transfer*.

It is easy to see that the transfer map is a k-linear map, but it is not a ring homomorphism. We will present some interesting properties of the transfer.

Proposition 2.1.1 ([13], Proposition 4.2.1). *If $H \leq K \leq G$ and $[G : H] < \infty$, then*

$$\mathrm{tr}_H^G = \mathrm{tr}_K^G \mathrm{tr}_H^K .$$

Proposition 2.1.2 ([13], Proposition 4.2.2). *If $H \leq G$ and $[G : H] < \infty$, then*

$$\mathrm{tr}_H^G \mathrm{res}_H^G = [G : H] \cdot id.$$

Theorem 2.1.3 ([10], Theorem 4.4.2). *Suppose that $H \leq G$ and $[G : H] < \infty$. Let A and B be $\mathrm{k}(G)$-modules. Then for $\zeta \in \mathrm{H}^*(G, A)$ and $\eta \in \mathrm{H}^*(H, B)$, we have*

$$\zeta \cdot \mathrm{tr}_H^G(\eta) = \mathrm{tr}_H^G(\mathrm{res}_H^G(\zeta) \cdot \eta).$$

Likewise,

$$\mathrm{tr}_H^G(\eta) \cdot \zeta = \mathrm{tr}_H^G(\eta \cdot \mathrm{res}_H^G(\zeta)).$$

Let k be a field or more general a noetherian ring. Consider $\mathrm{H}^*(H,k)$ as an $\mathrm{H}^*(G,k)$-module via the action given by $\zeta\eta = \mathrm{res}^G_H(\zeta)\cdot\eta$, for $\zeta \in \mathrm{H}^*(G,k)$ and $\eta \in \mathrm{H}^*(H,k)$. Then by Corollary 1.1.4, $\mathrm{H}^*(H,k)$ is a finitely generated $\mathrm{H}^*(G,k)$-module. From the above theorem, the transfer map tr^G_H is a homomorphism of $\mathrm{H}^*(G,k)$-modules. Hence, the transfer map can be determined by the image of a finite generating set of $\mathrm{H}^*(H,k)$ as an $\mathrm{H}^*(G,k)$-module.

Theorem 2.1.4 (Mackey formula)**.** *Let G be a group, H,K subgroups of G with $[G.H]<\infty$, and A a $\mathrm{k}(G)$-module. Then for $\xi \in \mathrm{H}^*(G,A)$ we have*

$$\mathrm{res}^G_K \mathrm{tr}^G_H(\xi) = \sum_{KxH} \mathrm{tr}^K_{K\cap xHx^{-1}} \mathrm{res}^{xHx^{-1}}_{K\cap xHx^{-1}}(x^*\xi)$$

where the sum is over the double cosets KxH.

Proof. See Theorem 3.5.4 in Carlson et al. [10]. □

The following results, which are helpful in testing the correctness of our implementation of the transfer, may be found in [1].

Proposition 2.1.5 ([1], Lemma 5.5)**.** *Let G,G' be groups and let $H\leq G$ with $[G:H]<\infty$. Then*

$$\mathrm{tr}^{G\times G'}_{H\times G'} = \mathrm{tr}^G_H \otimes id_{\mathrm{H}^*(G',k)}.$$

Proposition 2.1.6 ([1], Corollary 5.7)**.** *Let $G=\mathbb{Z}/p^n$ with $n>1$, $\{1\}<H<G$. Then we have*

(i). $\mathrm{tr}^G_H : \mathrm{H}^i(H,k) \longrightarrow \mathrm{H}^i(G,k)$ *is zero for i even and the identity map when i is odd,*

(ii). $\mathrm{res}^G_H : \mathrm{H}^i(G,k) \longrightarrow \mathrm{H}^i(H,k)$ *is zero for i odd and the identity map when i is even.*

Proposition 2.1.7 ([1], Corollary 5.9). *Let $G = (\mathbb{Z}/p)^n$ with $n > 0$, and suppose that $H < G$. Then*

$$\mathrm{tr}_H^G : \mathrm{H}^*(H, k) \longrightarrow \mathrm{H}^*(G, k)$$

is zero in all degrees.

2.2 The computation of transfer maps

For this section G is a finite p-group, H is a subgroup of G, and k is a finite field of characteristic p. We shall describe our implementation of the transfer map $\mathrm{tr}_H^G : \mathrm{H}^*(H, k) \to \mathrm{H}^*(G, k)$.

It is certainly that in the group cohomology computational projects with minimal resolutions approach, one may want to use the definition to calculate the transfer map. Of course, one needs to do some necessary slight changes to agree with computer performances. We shall adopt this method in our implementation.

Let T be a set of left coset representatives of H in G. Let $P_* \to k$ be a minimal $\mathrm{k}\,(G)$-projective resolution and $Q_* \to k$ be a minimal $\mathrm{k}\,(H)$-projective resolution. We consider P_* as a $\mathrm{k}\,(H)$-projective resolution of k. It follows from the comparison theorem that there exists a chain map $\Phi : P_* \to Q_*$ which is a lift of the identity map on k. For any cohomology element u in $\mathrm{H}^*(H, k) = \mathrm{Hom}_{\mathrm{k}(H)}(Q_*, k)$, we have $u \circ \Phi : P_* \to k$ a cocycle in $\mathrm{Hom}_{\mathrm{k}(H)}(P_*, k)$. By definition of the transfer, the trace of $u \circ \Phi$ given by

$$\mathrm{Tr}(u \circ \Phi)(x) = \sum_{t \in T} tu(\Phi(t^{-1}x))$$

presents the image of u by the transfer tr_H^G.

In our computation, k is always the finite field $\mathbb{F}_p$. For a p-group G of appropriate order, we may easily get a minimal resolution P_* of G and Q_* of its subgroup H from Green and King's group cohomology Sage package [18]. Notice that Green and King's package uses right modules for all of the calculations. It means that P_* and Q_* are right graded complexes. For convenience, we will often consider them as left modules in our argument however. Our implementation consists of the following main stages.

Step 1: Restricting a free $\mathbb{F}_p(G)$-module to a $\mathbb{F}_p(H)$-free module

Choose a set $T = \{t_1, \ldots, t_n\}$ of left coset representatives of H in G. We write $\mathbb{F}_p(G)$ as a free $\mathbb{F}_p(H)$-module

$$\mathbb{F}_p(G) = \bigoplus_{j=1}^{n} t_j \mathbb{F}_p(H).$$

Suppose that $M = \bigoplus_{i=1}^{r} e_i \mathbb{F}_p(G)$ is a right free $\mathbb{F}_p(G)$-module of rank r, then we can write

$$M = \bigoplus_{i=1}^{r} \bigoplus_{j=1}^{n} e_i t_j \mathbb{F}_p(H)$$

as right $\mathbb{F}_p(H)$-module of rank $r\,|T|$.

We implemented a routine called "unScramble" representing any element in the free $\mathbb{F}_p(G)$-module M on the basis $\{e_i t_j \mid 1 \leq i \leq r, 1 \leq j \leq n\}$ in the corresponding free $\mathbb{F}_p(H)$-module.

Step 2: Constructing the chain map

Now we will construct the chain map $\Phi : P_* \to Q_*$. For each $m \geq 0$, we write

$$P_m = \bigoplus_{i=1}^{\beta(m)} e_i \mathbb{F}_p(G) = \bigoplus_{i=1}^{\beta(m)} \bigoplus_{j=1}^{n} e_i t_j \mathbb{F}_p(H)$$

where $\beta(m)$ denotes the rank of P_m. The first chain map Φ_0 is easily seen to be

$$\Phi_0(u) = \sum_{i=1}^{r} \alpha_i,$$

where $u = \sum_{i=1}^{r} t_i \alpha_i, \alpha_i \in \mathbb{F}_p(H)$. Suppose that the mth-chain map Φ_m is known. We need to lift Φ_m to the map Φ_{m+1} such that the diagram

$$\begin{tikzcd} P_{m+1} = \bigoplus_{i=1}^{\beta(m+1)} \bigoplus_{j=1}^{n} e_i t_j \mathbb{F}_p(H) \arrow[r, "\partial_{m+1}"] \arrow[d, "\Phi_{m+1}"'] & P_m \arrow[d, "\Phi_m"] \\ Q_{m+1} \arrow[r, "d_{m+1}"'] & Q_m \end{tikzcd}$$

is commutative. The chain map Φ_{m+1} is characterized by the images of free generators

$$\{e_i t_j \mid 1 \leq i \leq \beta(m+1), 1 \leq j \leq n\}$$

of free $\mathbb{F}_p(H)$-module P_{m+1}. The image of $e_i t_j$ under Φ_{m+1} is given by setting

$$\Phi_{m+1}(e_i t_j) := \text{ a preimage of } \Phi_m(\partial_{m+1}(e_i) \cdot t_j) \text{ under } d_{m+1}.$$

Chapter 3

The Evens norm map

The purpose of this chapter is to describe a method to calculate the Evens norm map. The map was first invented by L. Evens in [11] and originally called the generalized transfer map (see [12]), because the properties of this map and the transfer are quite analogous. So as the transfer, the norm map is not a ring homomorphism in general, but it has several important applications. The first use of the norm is in Evens' proof of the finite generation of cohomology [11]. Another important application of the norm map is in the algebraic proof of Quillen's stratification theorem (see [10], p. 175). The norm also appeared in the work of Okuyama and Sasaki [23] on Serre's theorem with an attempt to give an improvement on Serre's bound, or of Minh [21] on upper bound of nilpotency degrees. However, the main application of the norm map that we want to consider in this thesis is to determine Steenrod operations on the cohomology rings of groups, which we will discuss in Chapter 4 in detail.

To begin this chapter, we shall introduce in Section 3.1 wreath product of groups and the method to build a projective resolution of the trivial module k over the group ring of this product. Then the cohomology ring of wreath products is described by a theorem of Nakaoka. The Section 3.2 is for the definition and properties of the norm map. In Section 3.3, we shall describe a method to calculate the norm map and our implementations.

The reader can find most of the material on the Evens norm map in the book by Carlson et al. [10]. We also refer the reader to the books by Benson [4] and Evens [13] for involved topics.

3.1 Cohomology of wreath products

Let S and H be groups and $\lambda : S \longrightarrow \mathrm{Aut}(H)$ an action of S on H. On the Cartesian product $H \times S$ define a multiplication by the rule

$$(h_1, \sigma_1)(h_2, \sigma_2) = (h_1 \lambda(\sigma_1)(h_2), \sigma_1 \sigma_2). \tag{3.1.1}$$

It is easy to show that $H \times S$ with the multiplication is a group. The group is called *semi-direct product* of H and S twisted by λ, and denoted by $H \rtimes_\lambda S$ or briefly by $H \rtimes S$ if λ is specified.

Let M be a $\mathrm{k}\,(H)$-projective module, and N a $\mathrm{k}\,(S)$-projective module. Suppose that there is an action of S on M such that

$$\sigma(hx) = \lambda(\sigma)(h)\sigma x \text{ for } \sigma \in S, h \in H, x \in M. \tag{3.1.2}$$

We define on $M \otimes_k N$ an action of $H \rtimes S$ by

$$(h, \sigma)(x \otimes y) = h(\sigma x) \otimes \sigma y. \tag{3.1.3}$$

Then we have

Proposition 3.1.1. *$M \otimes_k N$ is a projective $\mathrm{k}\,(H \rtimes S)$-module.*

Proof. See Proposition 2.5.1 in Evens [13]. $\square$

Now we can define a similar module structure for complexes. Let $\mathbf{X}_* \to k$ be a $\mathrm{k}\,(H)$-projective resolution, and $\mathbf{Y}_* \to k$ a $\mathrm{k}\,(S)$-projective resolution. Suppose that there is an action of S on $\mathbf{X}_*$ such that

(i). the augmentation and differentials are S-maps,

(ii). the action satisfies (3.1.2).

Then from Proposition 3.1.1, $\mathbf{X}_* \otimes_k \mathbf{Y}_* \to k$ with the action defined in (3.1.3) is a $H \rtimes S$-projective resolution.

Next we wish to consider the cohomology of the semi-direct product $H \rtimes S$. For a $H \rtimes S$-module A, let us consider the isomorphism of complexes

$$\mathrm{Hom}_{\mathrm{k}(H \rtimes S)}(\mathbf{X}_* \otimes_k \mathbf{Y}_*, A) \cong \mathrm{Hom}_{\mathrm{k}(S)}(\mathbf{Y}_*, \mathrm{Hom}_{\mathrm{k}(H)}(\mathbf{X}_*, A)). \tag{3.1.4}$$

The cohomology ring $\mathrm{H}^*(H \rtimes S, A)$ is known if we can calculate the cohomology of the term on the right hand side. In general, this work is quite complicated, except $\mathrm{Hom}_{\mathrm{k}(H)}(X, A)$ has trivial differential. The usually mentioned case for this situation is when k is a field, $\mathbf{X}_* \to k$ is a minimal projective resolution, and A is a simple $\mathrm{k}\,(S)$-module. In this case, we have

$$\mathrm{H}^*(H \rtimes S, A) \cong \mathrm{H}^*(S, \mathrm{Hom}_{\mathrm{k}(H)}(X, A)) = \mathrm{H}^*(S, \mathrm{H}^*(H, A)) \tag{3.1.5}$$

Definition 3.1.2. Let H be a group, $S \subseteq \mathcal{S}_n$ a subgroup of the symmetric group on n letters. Consider the action of S on the direct product H^n defined by

$$\sigma(h_1, \ldots, h_n) = (h_{\sigma^{-1}(1)}, \ldots, h_{\sigma^{-1}(n)}) \text{ for } \sigma \in S, h_i \in H. \tag{3.1.6}$$

Then the semi-direct product $H^n \rtimes S$ is called the *wreath product* of H by S and denoted by $H \wr S$. Explicitly, the multiplication in $H \wr S$ is given by

$$(h_1, \ldots, h_n; \sigma)(k_1, \ldots, k_n; \varsigma) = (h_1 k_{\sigma^{-1}(1)}, \ldots, h_n k_{\sigma^{-1}(n)}; \sigma\varsigma) \tag{3.1.7}$$

Let H be a finite group, $S \subseteq \mathcal{S}_n$ a subgroup of the symmetric group on n letters. To exploit the cohomology of a wreath product $H \wr S$, we discuss now the construction of a projective resolution over the wreath product from the given ones over H and S. Suppose that M is a $\mathrm{k}\,(H)$-module. Denote $M^{\otimes n} = \bigotimes_{i=1}^{n} M$. We can define on $M^{\otimes n}$ a $\mathrm{k}\,(H \wr S)$-module structure by letting

$$(h_1, \ldots, h_n; \sigma)(m_1 \otimes \cdots \otimes m_n) = h_{\sigma^{-1}(1)} m_{\sigma^{-1}(1)} \otimes \cdots \otimes h_{\sigma^{-1}(n)} m_{\sigma^{-1}(n)}.$$

To make the above construction for a complex, we need to introduce an appropriate sign because the action on the complex raises the permutation of factors in the tensor product. For a given $\mathrm{k}\,(H)$-complex $\mathbf{X}_*$, the action of $\mathrm{k}\,(H \wr S)$ on the n-fold tensor product complex $X^{\otimes n}$ is defined by

$$(h_1, \ldots, h_n; \sigma)(m_1 \otimes \cdots \otimes m_n) = (-1)^{\epsilon} h_{\sigma^{-1}(1)} m_{\sigma^{-1}(1)} \otimes \cdots \otimes h_{\sigma^{-1}(n)} m_{\sigma^{-1}(n)}, \tag{3.1.8}$$

where $\epsilon = \sum_{(i,j) \in N(\sigma)} \deg m_i \deg m_j$ and $N(\sigma) = \{(i,j) | i < j \text{ and } \sigma(i) > \sigma(j)\}$. It is tedious but quite routine to check that the action (3.1.8), in fact, satisfies the condition (i) and (ii) as above for the semi-direct product of H^n and S.

Let $\varepsilon : P_* \to k$ be a $\mathrm{k}\,(H)$-projective resolution, and $\eta : R_* \to k$ a $\mathrm{k}\,(S)$-projective resolution. By applying the Künneth formula, ε induces a $\mathrm{k}\,(H^n)$-projective resolution $\varepsilon^{\otimes n} : P_*^{\otimes n} \to k$. Hence as discussed above, $\varepsilon^{\otimes n} \otimes \eta : P_*^{\otimes n} \otimes_k R_* \to k$ is a $\mathrm{k}\,(H \wr S)$-projective resolution.

Lemma 3.1.3. *If $\varepsilon : P_* \to k$ is a minimal $\mathrm{k}\,(H)$-projective resolution and $\mu : Q_* \to k$ is a minimal $\mathrm{k}\,(K)$-projective resolution, then $\varepsilon \otimes \mu : P_* \otimes_k Q_* \to k \otimes_k k \cong k$ is a minimal $\mathrm{k}\,(H \times K)$-projective resolution.*

Proof. See the proof of Proposition 4.5.3 in Carlson et al. [10]. □

We can choose now $\varepsilon : P_* \to k$ to be a minimal $\mathrm{k}\,(H)$-projective resolution. Then we are in the good condition to apply (3.1.5) for $\mathbf{X} = P^{\otimes n}, \mathbf{Y} = R$, and hence we obtain the following theorem.

Theorem 3.1.4 (Nakaoka)**.** *Assume that k is a field. Suppose that H is a finite group, and S is a subgroup of the symmetric group $\mathcal{S}_n$ on n letters. Then*

$$\mathrm{H}^*(H \wr S, k) \cong \mathrm{H}^*(S, \mathrm{H}^*(H, k)^{\otimes n}),$$

and this is an isomorphism of rings.

Proof. For a proof of the theorem see Theorem 6.2.4 in Carlson et al. [10]. □

3.2 The Evens norm map

Let G be a finite group, and H a subgroups of G of index $[G : H] = n$. Let $T = \{t_1, \ldots, t_n\}$ be a set of representatives of the right cosets of H in G. We have then the *permutation representation* $\pi : G \to \mathcal{S}_n$ of G on T, where for each $g \in G, \pi(g)$ is defined by

$$t_i g = h_{i,g} t_{\pi(g)^{-1}(i)}, \tag{3.2.1}$$

for $1 \leq i \leq n$ and $h_{i,g} \in H$. Denote $S \subseteq \mathcal{S}_n$ the image of π. We define a map $\Phi : G \longrightarrow H \wr S$ by

$$\Phi(g) = (h_{1,g}, \ldots, h_{n,g}; \pi(g)). \tag{3.2.2}$$

It is easy to prove that Φ is a group monomorphism. This map is called the *monomial representation.*

Suppose M is a $\mathrm{k}\,(H)$-module (resp. complex). As in the previous section, $M^{\otimes n}$ has a module (resp. complex) structure over $\mathrm{k}\,(H \wr S)$. Thus we can make $M^{\otimes n}$ into a $\mathrm{k}\,(G)$-module (resp. complex) by restricting to G via the monomial representation $\Phi : G \longrightarrow H \wr S$. This $\mathrm{k}\,(G)$-module (resp. complex) is called the *tensor induced module (resp. complex)* from H to G, and we denote it by $M^{\otimes G/H}$. Especially, consider k as a trivial $\mathrm{k}\,(H)$-module then we obtain $k^{\otimes G/H} \cong k$.

For a given positive integer number d, we consider k as a $\mathrm{k}\,(H \wr S)$-module on which H^n acts trivially, S acts trivially if d is even, and if d is odd then S acts via the sign representation. We denote this module by $k^{(d)}$. Then the monomial representation Φ induces the restriction map

$$\Phi^* : \mathrm{H}^m(H \wr S, k^{(d)}) \longrightarrow \mathrm{H}^m(G, k^{(d)}). \tag{3.2.3}$$

Let $\varepsilon : P_* \to k$ be a $\mathrm{k}\,(H)$-projective resolution, and $\eta : R_* \to k$ a $\mathrm{k}\,(S)$-projective resolution. Suppose that $u \in \mathrm{H}^d(H, k)$ is represented by a cocycle $\alpha : P_d \to k$. In general, $\alpha^{\otimes n} : (P_d)^{\otimes n} \to k^{\otimes n}$ is not a homomorphism of $\mathrm{k}\,(H \wr S)$-modules, but $\alpha^{\otimes n} : (P_d)^{\otimes n} \to k^{(d)}$ is. Hence, we obtain a homomorphism of $\mathrm{k}\,(H \wr S)$-modules $\alpha^{\otimes n} \otimes \eta : (P^{\otimes n} \otimes_k R)_{dn} \to k^{(d)} \otimes k \cong k^{(d)}$. Moreover, we have

$$d^{\otimes}(\alpha^{\otimes n} \otimes \eta) = \sum_{i=1}^{n} (-1)^{(i-1)r} \alpha^{\otimes (i-1)} \otimes \delta(\alpha) \otimes \alpha^{\otimes (n-i)}.$$

Thus, if α is a cocycle then $d^{\otimes}(\alpha^{\otimes n} \otimes \eta) = 0$, hence $\alpha^{\otimes n} \otimes \eta$ is a cocycle. We define $u \wr 1 \in \mathrm{H}^{dn}(H \wr S, k^{(d)})$ to be the cohomology class represented by $\alpha^{\otimes n} \otimes \eta$. The following lemma tells us that the element $u \wr 1$ is well defined.

Lemma 3.2.1 (Steenrod)**.** *Suppose that C_* is a complex of $\mathrm{k}\,(H)$-modules and that $\alpha, \beta : P_* \longrightarrow C_*$, are chain maps that are chain homotopic. Then the chain of the n-fold tensor product*

$$\alpha^{\otimes n} \otimes \eta, \beta^{\otimes n} \otimes \eta : (P^{\otimes n} \otimes_k R)_* \longrightarrow C_*^{\otimes n}$$

are also chain homotopic as chain maps of $\mathrm{k}\,(H \wr S)$-complexes.

Proof. For a proof see Lemma 6.3.1 in Carlson et al. [10]. □

Now we define the Evens norm map as follows.

Definition 3.2.2. For any degree d, the *Evens norm map*

$$\mathcal{N}orm_H^G : \mathrm{H}^d(H, k) \longrightarrow \mathrm{H}^{dn}(G, k^{(d)})$$

is given by $\mathcal{N}orm_H^G(u) = \Phi^*(u \wr 1)$.

Notice that $k^{(d)} \cong k$ as an isomorphism of $\mathrm{k}\,(H^n)$-modules, but as $\mathrm{k}\,(H \wr S)$-modules, it is not the case because of the sign. Therefore, we usually restrict ourself to the cases where the sign is not an issue. The most interesting cases are:

(i). d is even,

(ii). d is odd, $p = char(k)$ is odd and S is generated by a cycle of length p,

(iii). $char(k) = 2$.

In these cases, we have the norm map

$$\mathcal{N}orm_H^G : \mathrm{H}^d(H, k) \longrightarrow \mathrm{H}^{dn}(G, k).$$

Theorem 3.2.3. *Suppose that H is a subgroup of G of index $[G : H] = n$. The norm map has the following properties:*

(i). If K is a subgroup of G such that $H \leq K \leq G$, and $u \in \mathrm{H}^r(H, k)$ then

$$\mathcal{N}orm_K^G(\mathcal{N}orm_H^K(u)) = \mathcal{N}orm_H^G(u).$$

(ii). If $u \in \mathrm{H}^r(H, k)$ and $v \in \mathrm{H}^s(H, k)$ then

$$\mathcal{N}orm_H^G(uv) = (-1)^{rsn(n-1)/2}\mathcal{N}orm_H^G(u)\mathcal{N}orm_H^G(v).$$

(iii). If K is a subgroup of G and $u \in \mathrm{H}^r(H, k)$. Then

$$\mathrm{res}_G^K(\mathcal{N}orm_H^G(u)) = \prod_{KxH} \mathcal{N}orm_{K\cap xHx^{-1}}^K(\mathrm{res}_{K\cap xHx^{-1}}^{xHx^{-1}}(x^*(u)))$$

where the product is over a set of representatives of the KxH-double cosets in G.

(iv). If $H \trianglelefteq G$, then $\mathrm{res}_H^G(\mathcal{N}orm_H^G(u)) = \prod_{x\in G/H} x^*(u)$.

(v). If $u, v \in \mathrm{H}^r(H, k)$, then $\mathcal{N}orm_H^G(u + v) = \mathcal{N}orm_H^G(u) + \mu + \mathcal{N}orm_H^G(v)$ where μ is a sum of transfer from proper subgroups of G. If $H \lhd G$, then all of the subgroups contain H.

(vi). $\mathcal{N}orm_H^G(1 + u) = 1 + \mathrm{tr}_H^G(u) + \ldots + \mathcal{N}orm_H^G(u)$.

Proof. See Theorem 6.3.5 in Carlson et al. [10]. □

3.3 The computation of norm maps

In this section, G is a finite p-group and H is a subgroup of G of index $[G : H] = n$. We will always restrict ourselves to the cases as in the previous section, where the sign is not a matter. Then the construction of the norm map

$$\mathcal{N}orm_H^G : \mathrm{H}^*(H,k) \longrightarrow \mathrm{H}^*(G,k)$$

is as follows.

Notice that we will adopt in this section the constructions of minimal resolutions from the Sage package of Green and King, where right modules are used. Hence, unless otherwise stated, we assume throughout this section that modules are right modules. Furthermore, we also need to make some subtle adjustments in definitions in the previous sections necessary to fit the situation that actions are from the right hand side. Namely, let $T = \{t_1, \dots, t_n\}$ be a set of representatives of the left cosets (instead of the right cosets as in Section 3.2) of H in G. The permutation representation $\pi : G \to \mathfrak{S}_n$ is redefined by

$$gt_i = t_{\pi(g)(i)}h_{i,g}. \tag{3.3.1}$$

Keeping the notations of S and Φ as in the first part of Section 3.2. The wreath product defined in Definition 3.1.2 is modified by redefining the multiplication (3.1.7) by

$$(h_1, \dots, h_n; \sigma)(k_1, \dots, k_n; \varsigma) = (h_{\varsigma(1)}k_1, \dots, h_{\varsigma(n)}k_n; \sigma\varsigma). \tag{3.3.2}$$

Let $\varepsilon : P_* \to k$ be a $\mathrm{k}\,(H)$-projective resolution, and $\eta : R_* \to k$ a $\mathrm{k}\,(S)$-projective resolution. Then as discussed in Section 3.1, $\varepsilon^{\otimes n} \otimes \eta : P_*^{\otimes n} \otimes_k R_* \to k$ is a $\mathrm{k}\,(H \wr S)$-projective resolution. Notice that the action (3.1.8) of $\mathrm{k}\,(H \wr S)$ on the n-fold tensor product complex $P^{\otimes n}$ is now modified to be

$$(m_1 \otimes \cdots \otimes m_n)(h_1, \dots, h_n; \sigma) = (-1)^{\epsilon} m_{\sigma(1)}h_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(n)}h_{\sigma(n)}. \tag{3.3.3}$$

We make $\varepsilon^{\otimes n} \otimes \eta : P_*^{\otimes n} \otimes_k R_* \to k$ into a $\mathrm{k}\,(G)$-projective resolution by restricting to G via the monomorphism Φ. Suppose now that $\kappa : Q_* \to k$ is a $\mathrm{k}\,(G)$-projective resolution. It follows from the comparison theorem that there exists a chain map $\Omega : Q_* \to (P^{\otimes n} \otimes_k R)_*$ which is a lift of the identity map on k.

For an element $u \in \mathrm{H}^d(H, k)$ represented by a cocycle $\alpha : P_d \to k$, the tensor $\alpha^{\otimes n} \otimes \eta : (P^{\otimes n} \otimes_k R)_{dn} \to k$ is also a cocycle. Then the composition with Ω yields a cocycle $(\alpha^{\otimes n} \otimes \eta) \circ \Omega : Q_{dn} \to k$. Hence, the image of u by the norm map is $\mathcal{N}orm_H^G(u) = [(\alpha^{\otimes n} \otimes \eta) \circ \Omega] \in \mathrm{H}^{dn}(G, k)$.

Base on the method described above, the computation of the norm map consists of the following main stages.

Step 1: Constructing the tensor product of resolutions

For a given family $\{P_*^i \mid i = 1, \ldots, r\}$ of (minimal) $\mathrm{k}(G_i)$-projective resolutions of k. The tensor product $P_* = P_*^1 \otimes_k P_*^2 \otimes_k \cdots \otimes_k P_*^r$ is a (minimal) $\mathrm{k}(G_1 \times \cdots \times G_r)$-projective resolutions of k with the differentials given by

$$d(m_1 \otimes m_2 \otimes \cdots \otimes m_r) = \sum_{i=1}^{r} (-1)^{\sum_{j=1}^{i-1} |m_j|} m_1 \otimes \cdots \otimes \partial(m_i) \otimes \cdots \otimes m_r. \qquad (3.3.4)$$

The family $\{P_*^i \mid i = 1, \ldots, r\}$ of minimal resolutions is constructed by using Green and King's Sage package. However, we have to emphasize here that we will not use such method to construct the tensor product P_* of the given resolutions in this step, although it might be the minimal resolution of a p-group. We actually want to build a resolution P_* so that we could be able to apply the action (3.3.3). Note that for each m, P_m is a free $\mathrm{k}(G_1 \times G_2 \times \cdots \times G_r)$-module. Denote the rank of P_m by $\beta(m)$. The group algebra $\mathrm{k}(G_1 \times G_2 \times \cdots \times G_r)$ is created by using a tool which is written by David Green in C-code, and then wrapped in Sage by Simon King (see [18]). The m^{th} differential $d_m : P_m \to P_{m-1}$ is calculated by using (3.3.4), and is characterized by the images of $\beta(m)$ free generators of P_m. Additionally, the tensor product resolution is built in such a way that we could be able to calculate the tensor product $m_1 \otimes m_2 \otimes \cdots \otimes m_r$, as well as to decompose a tensor into components. Moreover, to build a chain map Ω as above, we need to compute the preimages by the differentials. An efficient tool was developed by Green [15] in C-code using noncommutative Gröbner basis method is used. Note that a linear algebra method could be easily used here. But the Gröbner basis method is the best choice because it works linearly over group rings, rather than linearly over the ground field k as in linear algebra method. The output data from this tool are stored in .ugb-files.

Step 2: Building the resolution of wreath product

Suppose now that G is a finite p-group and H a subgroup of G of index $[G : H] = n$. We will determine the group S and the monomial representation Φ first. If S is a p-group, then the minimal projective resolution R_* of the trivial $\mathrm{k}\,(S)$-module k is constructed in Green and King's package. Let P_* be the minimal $\mathrm{k}\,(H)$-projective resolution of k. Then the tensor product $P_*^{\otimes n} \otimes_k R_*$ is created and stored as a $\mathrm{k}\,(H^n \times S)$-projective resolution by the method in Step 1. Next, we will determine the action of $\mathrm{k}\,(H \wr S)$ on $P_*^{\otimes n} \otimes_k R_*$ satisfying (3.3.3). Notice that for the mentioned reason in Step 1, we will not construct the $\mathrm{k}\,(H^n \rtimes S)$-projective resolution of k directly in our method. Finally, the action of G is determined via the map Φ.

Step 3: Computing the chain map Ω

Let Q_* be a minimal $\mathrm{k}\,(G)$-projective resolution of k. The first chain map $\Omega_0 : Q_0 \to (P^{\otimes n} \otimes_k R)_0$ is easy to write down. Suppose that the m^{th} chain map Ω_m is known.

We need to lift Ω_m to the map Ω_{m+1} such that the diagram

$$\begin{tikzcd}
Q_{m+1} \arrow[r, "\partial_{m+1}"] \arrow[d, "\Omega_{m+1}"'] & Q_m \arrow[d, "\Omega_m"] \\
(P^{\otimes n} \otimes_k R)_{m+1} \arrow[r, "d_{m+1}"'] & (P^{\otimes n} \otimes_k R)_m
\end{tikzcd}$$

is commutative. Let u be an arbitrary element in Q_{m+1}. The image of u under Ω_{m+1} is given by setting

$$\Omega_{m+1}(u) := \text{ a preimage of } \Omega_m \partial_{m+1}(u) \text{ under } d_{m+1}.$$

Computing the preimages is easily done by the data we got in the first step. In fact, the chain map Ω_{m+1} is characterized by the images of free generators of Q_{m+1}.

Chapter 4

Steenrod operations

In this chapter we shall develop a method to calculate the Steenrod operations on the mod-2 cohomology ring of finite groups. In general, the algebra of these operations is a great important structure occurring in the theory of cohomology operations, which plays an important role in algebraic topology and has many important applications. In particular, many calculations in cohomology theory of groups have relied on the Steenrod algebra.

The aim of this chapter is to give a computational treatment to compute Steenrod operations on cohomology rings of as many groups as possible. There are some available approaches for computing Steenrod operations on the cohomology rings of 2-groups (see Rusin [25] and Guillot [19]). However, their methods work only for some limited cases. In Section 4.2, we will introduce a method to compute Steenrod operations for several finite groups effectively. Section 4.1 is for a brief introduction of Steenrod algebra. We will introduce in this section the construction of Steenrod operations on cohomology rings by using the Evens norm maps.

We usually consider the monograph by Steenrod and Epstein [26] as a standard reference for the topological approach to the Steenrod algebra. We also refer the reader to [22] for the applications of Steenrod operations to homotopy theory. Most of the material of Steenrod algebra in this chapter may be found in the books by Benson [4] and Carlson et al. [10].

4.1 The Steenrod algebra

Throughout this chapter, we shall restrict ourselves to the computation of mod-2 Steenrod operations. For simplicity, we will introduce here the basic knowledge of the mod-2 Steenrod algebra.

Definition 4.1.1. The mod-2 Steenrod algebra is the quotient

$$\mathcal{A}_2 = \mathbb{F}_2\langle Sq^1, Sq^2, Sq^3, \ldots\rangle/\mathcal{J}$$

where $\mathcal{J}$ is the ideal generated by Adem relations

$$Sq^r Sq^s = \sum_{j=0}^{[r/2]} \binom{s-1-j}{r-2j} Sq^{r+s-j} Sq^j.$$

for all r and s with $0 < r < 2s$. Here $Sq^0 = 1$ and $Sq^i = 0$ for $i < 0$.

Theorem 4.1.2. $\{Sq^{2^i} | i \geq 0\}$ *generate $\mathcal{A}_2$ as an algebra.*

Proof. For a proof see Theorem 4.3 in Steenrod and Epstein [26]. □

It will be more convenient for our computations of mod-2 Steenrod operations when we use the Wall basis which is defined as following (see [24]).

Definition 4.1.3. For integers $m \geq k$, let

$$Q_k^m = Sq^{2^k} Sq^{2^{k+1}} \ldots Sq^{2^m}.$$

The set of words in the Q_k^m's, ordered in decreasing left lexicographic order on the pairs (m, k), is a basis for $\mathcal{A}_2$, called Wall basis.

Let G be a finite group, and C_2 a cyclic group of order 2. Then we have $\mathrm{H}^*(C_2, \mathbb{F}_2) = \mathbb{F}_2[u]$, where u is the generator of $\mathrm{H}^1(C_2, \mathbb{F}_2)$. We consider the norm map $\mathcal{N}orm_G^{G\times C_2} : \mathrm{H}^*(G, \mathbb{F}_2) \to \mathrm{H}^*(G \times C_2, \mathbb{F}_2)$.

Definition 4.1.4. For $x \in \mathrm{H}^r(G, \mathbb{F}_2)$, define $D_i(x) \in \mathrm{H}^{2r-i}(G, \mathbb{F}_2)$ with $0 \leq i \leq 2r$ by the formula

$$Norm_G^{G\times C_2}(x) = \sum_{i=0}^{2r} D_i(x) \otimes u^i.$$

Then the action of Steenrod operations in the Steenrod algebra $\mathcal{A}_2$ on the cohomology ring $\mathrm{H}^*(G, \mathbb{F}_2)$ is defined to be the homomorphism $Sq^i : \mathrm{H}^r(G, \mathbb{F}_2) \to \mathrm{H}^{r+i}(G, \mathbb{F}_2)$ given by $Sq^i(x) = D_{r-i}(x)$.

Let G be a finite group. Recall that the Bockstein homomorphism

$$\beta : \mathrm{H}^r(G, \mathbb{F}_p) \to \mathrm{H}^{r+1}(G, \mathbb{F}_p)$$

is the connecting homomorphism in the long exact sequence associated to the short exact sequence

$$0 \to \mathbb{F}_p \xrightarrow{p} \mathbb{Z}/p^2 \to \mathbb{F}_p \to 0$$

of coefficients. The action of Steenrod operations on the cohomology of G has the following properties.

Theorem 4.1.5. *The cohomology ring* $\mathrm{H}^*(G, \mathbb{F}_2)$ *is a module over the mod-*2 *Steenrod algebra* $\mathcal{A}_2$*. In addition, the action of the operations has the following properties.*

(i). The additive homomorphism $Sq^i : \mathrm{H}^r(G, \mathbb{F}_2) \to \mathrm{H}^{r+i}(G, \mathbb{F}_2)$ *are natural transformation of functors for all* i *and* r*.*

(ii). $Sq^0 = 1$.

(iii). *If* $\deg x = r$ *then* $Sq^r(x) = x^2$.

(iv). *If* $\deg x < i$ *then* $Sq^i(x) = 0$.

(v). $Sq^1 = \beta : \mathrm{H}^r(G, \mathbb{F}_2) \to \mathrm{H}^{r+1}(G, \mathbb{F}_2)$ *the Bockstein homomorphism.*

(vi). *(Cartan formula)*

$$Sq^k(xy) = \sum_{j=0}^{k} Sq^j(x)Sq^{k-j}(y).$$

Proof. See Section 4.5 in Benson [4] for a detailed proof. $\square$

4.2 The computation of Steenrod operations

In this section, we will describe the algorithm to calculate the action of the mod-2 Steenrod algebra $\mathcal{A}_2$ on the mod-2 cohomology rings of small groups. The cohomology rings, as elementary material for our computations, are from the Sage package of D. Green and S. King [18]. As in [18], the modular cohomology rings of all 2-groups of orders up to 128 have been calculated. Also, the cohomology of many non-prime power groups were computed by reducing to the Hall subgroups via the stable element method. Our goal in this project is to compute the Steenrod operations on the cohomology rings of as many groups as possible. Recall that the notation $SmallGroup(n, r)$ is used to denotes a group of order n and number r in the small group library of the computer algebra system GAP.

4.2.1 Sufficient condition and recursive calculation

Let G be a finite group. To determine the action of the mod-2 Steenrod algebra $\mathcal{A}_2$ on the cohomology ring $\mathrm{H}^*(G, \mathbb{F}_2)$, it is sufficient to compute the basic actions

$$Sq^{2^i}(u), \forall i \geq 0, 2^i < \deg u, \tag{4.2.1}$$

for each generator u of $\mathrm{H}^*(G, \mathbb{F}_2)$.

Indeed, suppose that $\mathcal{G}$ is a generating set of the cohomology ring $\mathrm{H}^*(G, \mathbb{F}_2)$. Let St be an arbitrary Steenrod operation in $\mathcal{A}_2$, and u an element in $\mathrm{H}^*(G, \mathbb{F}_2)$. By Theorem 4.1.2 and by induction, the computation of $St(u)$ may be reduced to the case of $Sq^{2^k}(u_1 u_2 \cdots u_n)$, where $u_i \in \mathcal{G}$. A general form of the Cartan formula in Theorem 4.1.5 is

$$Sq^k(u_1 u_2 \cdots u_n) = \sum_{s_1+s_2+\ldots+s_n=k} Sq^{s_1}(u_1) Sq^{s_2}(u_2) \cdots Sq^{s_n}(u_n). \tag{4.2.2}$$

Apply again Theorem 4.1.2 for the right hand side of the Cartan formula, we may reduce our computation again to the cases of $Sq^{2^s}(u_i), u_i \in \mathcal{G}$.

We suppose now that all basic actions in (4.2.1) have been calculated somewhere and stored in a cache. We want to compute the action of an arbitrary Steenrod operation St on any element $u \in \mathrm{H}^*(G, \mathbb{F}_2)$ by the following recursive algorithm.

Algorithm 4.2.1. (**SteenrodOp**)

Input: A Steenrod operation St, a homogeneous element $u \in \mathrm{H}^*(G, \mathbb{F}_2)$.

Output: The action St(u).

if $St = 0$ or $u = 0$ **then**

  return 0

if $St = Sq^0$ **then**

return u

if $St = Sq^r$ **then**

if $\deg u < r$ **then**

return 0

else if $\deg u = r$ **then**

return u^2

Represent St as a sum of generators in the Wall basis: $St := \sum_{s,t} Q^s_t$.

Represent u over the generating set: $u := \sum u_{i_1} u_{i_2} \cdots u_{i_n}$ where $u_{i_j} \in \mathcal{G}$.

if $St = Sq^{2^{k_1}} Sq^{2^{k_2}} \cdots Sq^{2^{k_m}}$ and $u = u_{i_1} u_{i_2} \cdots u_{i_n}$ **then**

if $m = 1$ and $n = 1$ **then**

Find $Sq^{2^{k_1}}(u_{i_1})$ from the cache.

return $Sq^{2^{k_1}}(u_{i_1})$

else if $m = 1$ and $n > 1$ **then**

#Apply the formula 4.2.2.

return $\sum_{s_1+s_2+\ldots+s_n=2^{k_1}} \prod_{i=1}^n$ **SteenrodOp**(Sq^{s_i}, u_i)

else

$out :=$ **SteenrodOp**$(Sq^{2^{k_m}}, u)$

for $i = m-1$ to 1 **do**

$out :=$ **SteenrodOp**$(Sq^{2^{k_i}}, out)$

return out

else

return $\sum$ **SteenrodOp**$(Q^s_t, u_{i_1} u_{i_2} \cdots u_{i_n})$

In our implementation, we use the available Sage package of Palmieri on Steenrod algebra to treat the Adem relations and the Wall basis.

4.2.2 Detection method

Definition 4.2.2. Let G be a finite group, and $\mathcal{H}$ a family of proper subgroups of G. We say that $\mathcal{H}$ detects the cohomology $\mathrm{H}^*(G,k)$ if the product of the restriction maps

$$\prod_{H\in\mathcal{H}} \mathrm{res}^G_H : \mathrm{H}^*(G,k) \to \prod_{H\in\mathcal{H}} \mathrm{H}^*(H,k) \tag{4.2.3}$$

is an injection.

Notice that in the case k is a field of characteristic p, if P is a Sylow p-subgroup of G then by Proposition 2.1.2 the restriction res^G_P is an injection. Hence, P detects the cohomology ring $\mathrm{H}^*(G,k)$.

Examples.

1. The cohomology of the Mathieu groups M_{22} and M_{23} are detected on their Sylow 2-subgroups P, where $P \cong SmallGroup(128, 931)$.

2. The cohomology of the Modular group Mod_{16} of order 16 is undetectable, it means that the essential ideal $Ess(Mod_{16})$ which is defined to be the kernel of the product of the restriction maps to proper subgroups is not trivial.

Proposition 4.2.3. *Suppose that* $\mathrm{H}^*(G,k)$ *has depth* d*. Then* $\mathrm{H}^*(G,k)$ *is detected on the set* $\mathcal{H}_d$ *of all centralizers of elementary abelian* p*-subgroups of rank* d*.*

Proof. See Corollary 12.5.3 in Carlson et al. [10]. □

Corrolary 4.2.4. *The cohomology of the Mathieu group* M_{22} *and its Sylow* 2*-subgroups are detected on the* 2*-subgroups of order* 16*.*

Proof. The computation of Green and King in [18] tells us that the depth of the cohomology ring of Mathieu group M_{22} is 2. There are four conjugacy classes elementary abelian 2-subgroups of rank 2 in M_{22}. The centralizers of these groups are

$K_1 \cong \mathit{SmallGroup}(64,242), K_2 \cong \mathit{SmallGroup}(48,242), K_3 \cong \mathit{SmallGroup}(32,27)$ and $K_4 \cong \mathit{SmallGroup}(16,8)$ in the small group library of the computer algebra system GAP. Using publicly available essential ideal computations by Green in [14], it is easy to see that K_1 is detected on subgroups of order 32 which are isomorphic to $\mathit{SmallGroup}(32,27)$ and $\mathit{SmallGroup}(32,31)$. In turn, these groups are detected on subgroups of order 16. K_2 is detected on its Sylow 2 subgroup of order 16. By using GAP, we can easily find this collection of subgroups of order 16 which detects M_{22}. $\square$

Now we use the detection to compute Steenrod operations. Let G be a finite 2-group. We have the following cases.

Case 1: The cohomology ring $\mathrm{H}^*(G,\mathbb{F}_2)$ is detectable. It means that we can find out a family $\mathcal{H}$ of proper subgroups of G detecting $\mathrm{H}^*(G,\mathbb{F}_2)$. Then we may consider $\mathrm{H}^*(G,\mathbb{F}_2)$ as a subring of $\prod\limits_{H\in\mathcal{H}} \mathrm{H}^*(H,\mathbb{F}_2)$ due to the injection $\prod\limits_{H\in\mathcal{H}} \mathrm{res}^G_H$. Suppose that the Steenrod squares on $\mathrm{H}^*(H,\mathbb{F}_2), \forall H\in\mathcal{H}$ have been computed. Then the Steenrod squares on $\mathrm{H}^*(G,\mathbb{F}_2)$ are determined.

Case 2: The cohomology ring $\mathrm{H}^*(G,\mathbb{F}_2)$ is undetectable. Suppose that $\mathcal{H}$ is a family of proper subgroups of G such that the Steenrod squares on $\mathrm{H}^*(H,\mathbb{F}_2), \forall H\in\mathcal{H}$ have been computed. For each Steenrod square Sq^{2^k} and each element u in the generating set $\mathcal{G}$ of $\mathrm{H}^*(G,\mathbb{F}_2)$, we have

$$\prod_{H\in\mathcal{H}} \mathrm{res}^G_H(Sq^{2^k}(u)) = \prod_{H\in\mathcal{H}} Sq^{2^k}(\mathrm{res}^G_H(u)).$$

Notice that $\mathcal{H}$ does not detect $\mathrm{H}^*(G,\mathbb{F}_2)$, so the map $\prod\limits_{H\in\mathcal{H}} \mathrm{res}^G_H$ is not injective. Suppose that the map $\prod\limits_{H\in\mathcal{H}} \mathrm{res}^G_H$ produces a set $\mathcal{E}$ of candidates for $Sq^{2^k}(u)$ with $\#(\mathcal{E}) > 1$. We will try to determine the right element from the set $\mathcal{E}$ by using an appropriate "filter". We will discuss these "filters" in the next sections in details.

Examples. Consider the Modular group $G = Mod_{16}$ of order 16. From the computations in [18], the generators of $\mathrm{H}^*(G, \mathbb{F}_2)$ are a_1, b_1, c_3, d_4, where the index in each generator indicates the degree of the element. The basic actions that we want to compute on the cohomology ring of this group are: $Sq^1(c_3), Sq^2(c_3), Sq^1(d_4), Sq^2(d_4)$. The Steenrod operations on the cohomology rings of all groups of order 8 were calculated already. Then the detection method gives us that

$$\begin{aligned} Sq^1(c_3) &= 0, \\ Sq^2(c_3) &= b_1^2 c_3 + d_4 a_1, \\ Sq^1(d_4) &= b_1^2 c_3, \end{aligned}$$

and there remains an undetermined action $Sq^2(d_4)$ with the set of candidates $\mathcal{E} = \{d_4 b_1^2, d_4 b_1^2 + d_4 a_1 b_1\}$.

Implementation:

The main work in this stage is to calculate the preimages of a family $\{u_H \mid H \in \mathcal{H}\} \in \prod\limits_{H \in \mathcal{H}} \mathrm{H}^*(H, k)$ under the map $\prod\limits_{H \in \mathcal{H}} \mathrm{res}^G_H$.

We choose linear algebra approach for our implementation. For each degree r, the restriction map res^G_H is a linear map from $\mathrm{H}^r(G, \mathbb{F}_2)$ to $\mathrm{H}^r(H, \mathbb{F}_2)$. In general, suppose that $\{f_i : V \to V_i \mid i = 1, \ldots, n\}$ is a family of homomorphisms of vector spaces with matrices $\{A_i\}_i$ respectively. Then the product map $\prod_{i=1}^n f_i : V \to \prod_{i=1}^n V_i$ is a linear map with matrix

$$A = (A_1 \mid A_2 \mid \ldots \mid A_n).$$

We solve the linear equation $xA = b$ to find the preimages of $b \in \prod_{i=1}^n V_i$ under the linear map $\prod_{i=1}^n f_i$.

The restriction map has been implemented in the Green and King's Sage package. We use the C-MeatAxe wrapper in the mentioned package to work with matrices.

4.2.3 Comodule map as a "filter"

Let G be a finite p-group, and C its greatest central elementary abelian subgroup. Define a group homomorphism $\mu : G \times C \to G$ by $\mu(g, c) = g \cdot c$. Then the induced map

$$\mu^* : \mathrm{H}^*(G, \mathbb{F}_p) \to \mathrm{H}^*(G, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{H}^*(C, \mathbb{F}_p) \tag{4.2.4}$$

is a ring homomorphism. As in [6], this also means that the group multiplication μ induces a $\mathrm{H}^*(C, \mathbb{F}_p)$-comodule structure on $\mathrm{H}^*(G, \mathbb{F}_p)$, with the comodule map μ^*.

Now we concentrate on the case G is a 2-group. Denote by r the 2-rank of C. Let $u_1, \dots, u_r \in \mathrm{H}^1(C, \mathbb{F}_2)$ be the generators of the cohomology ring $\mathrm{H}^*(C, \mathbb{F}_2)$. For any $x \in \mathrm{H}^n(G, \mathbb{F}_2), n > 0$, we have

$$\mu^*(x) = \sum_{i_1 + \ldots + i_r + s_{i_1, \ldots, i_r} = n} x_{i_1, \ldots, i_r} \otimes u_1^{i_1} \cdots u_r^{i_r}, \tag{4.2.5}$$

where $x_{i_1, \ldots, i_r} \in \mathrm{H}^{s_{i_1, \ldots, i_r}}(G, \mathbb{F}_2)$.

Let $\pi_1 : G \times C \to G$ be the projection given by $\pi_1(g, c) = g$, and $\pi_2 : G \times C \to C$ the projection given by $\pi_2(g, c) = c$. We may view $\mathrm{H}^*(G, \mathbb{F}_2) \otimes 1$ as the image of the induced map π_1^*, and $1 \otimes \mathrm{H}^*(C, \mathbb{F}_2)$ as the image of the induced map π_2^*. Let $i_1 : G \to G \times C, i_1(g) = (g, 1)$ and $i_2 : C \to G \times C, i_2(c) = (1, c)$ the embeddings to $G \times C$. From the fact that the composition

$$C \xrightarrow{i_2} G \times C \xrightarrow{\mu} G$$

is the inclusion of C into G, we have $\mathrm{res}^{G\times C}_C \circ \mu^* = \mathrm{res}^G_C$. It follows that

$$\mathrm{res}^G_C(x) = \sum_{i_1+\ldots+i_r=n} a_{i_1,\ldots,i_r} u_1^{i_1}\cdots u_r^{i_r}$$

with $a_{i_1,\ldots,i_r} \in \mathbb{F}_2$. Furthermore, since the composition

$$G \xrightarrow{i_1} G\times C \xrightarrow{\mu} G$$

is the identity map in G, we have $\mathrm{res}^{G\times C}_G \circ \mu^* = id_{\mathrm{H}^*(G,\mathbb{F}_2)}$. So there exits $x' \in \mathrm{H}^n(G,\mathbb{F}_2)$ such that $\mathrm{res}^{G\times C}_G(x'\otimes 1) = x$. Moreover, as $\pi_1 \circ i_1 = id_G$, we have $\mathrm{res}^{G\times C}_G \circ \pi_1^* = id_{\mathrm{H}^*(G,\mathbb{F}_2)}$. It follows that $x' = x$. Hence, we can write

$$\mu^*(x) = x\otimes 1 + 1\otimes \mathrm{res}^G_C(x) + \text{other terms}. \tag{4.2.6}$$

Definition 4.2.5. An element $x \in \mathrm{H}^*(G,\mathbb{F}_2)$ is called *primitive* if

$$\mu^*(x) = x\otimes 1 \in \mathrm{H}^*(G,\mathbb{F}_2)\otimes_{\mathbb{F}_2} \mathrm{H}^*(C,\mathbb{F}_2).$$

Now we discuss how to use the comodule map to compute Steenrod operations. Suppose that we are in the mentioned case as in the previous section, where $\mathrm{H}^*(G,\mathbb{F}_2)$ is undetectable and $\mathcal{H}$ is a family of proper subgroups of G such that the Steenrod squares are computed. For a Steenrod square Sq^{2^k} and an element u in the generating set $\mathcal{G}$ of $\mathrm{H}^*(G,\mathbb{F}_2)$, suppose that $\prod_{H\in\mathcal{H}} \mathrm{res}^G_H$ produces a set $\mathcal{E}$ of candidates for $Sq^{2^k}(u)$ with $\#(\mathcal{E}) > 1$. By applying (4.2.6), we have

$$\begin{aligned}\mu^*(Sq^{2^k}(u)) &= Sq^{2^k}(\mu^*(u))\\ &= Sq^{2^k}(u)\otimes 1 + 1\otimes Sq^{2^k}(\mathrm{res}^G_C(u)) + Sq^{2^k}(\text{other terms}).\end{aligned} \tag{4.2.7}$$

By considering the right hand side of the formula as a polynomial in r variables $u_1,\ldots,u_r$, we compare it with $\mu^*(x)$ for $x \in \mathcal{E}$ to determine the correct element. From (4.2.7), it is easy to see that we may use Sq^{2^k}(other terms) in the right hand side of (4.2.7) as a criterion for the comparison. Unfortunately, in many cases Sq^{2^k}(other terms) is undetermined, hence it causes an difficulty in implementing the comparisons. Therefore, we shall change a bit the comparison method for the sake of the implementation.

Suppose that $u_1, \dots, u_n$ are elements in $\mathrm{H}^*(G, \mathbb{F}_2)$ such that the actions $Sq^{k_1}(u_1)$, $Sq^{k_2}(u_2), \dots, Sq^{k_n}(u_n)$ are not determined yet. Let $\mathcal{E}_1, \dots, \mathcal{E}_n$ be the sets of candidates for the above actions respectively. Set $\mathcal{X} = \prod_{i=1}^n \mathcal{E}_i$. The algorithm for the comparisons is as follows.

Algorithm 4.2.6. (Comparison)

for $(x_1, \dots, x_n) \in \mathcal{X}$ **do**

  Assume $Sq^{k_i}(u_i) := x_i$ for $i = 1, \dots, n$.

  Using the Cartan formula in (4.2.11) to calculate the Steenrod operations on $\mathrm{H}^*(G \times C, \mathbb{F}_2)$.

  for i from 1 to n **do**

    Compute $A = \mu^*(Sq^{k_i}(u_i))$.

    Compute $B = Sq^{2^k}(\mu^*(u_i))$.

    if $A \neq B$ **then**

      break

    else

      Add $(x_1, \dots, x_n)$ into a list $\mathcal{L}$.

if $\mathcal{L} = \{(x_1^0, \dots, x_n^0)\}$ **then**

  return $Sq^{k_i}(u_i) := x_i^0$ for $i = 1, \dots, n$.

else

  Find the sets $\mathcal{F}_1 \subseteq \mathcal{E}_1, \dots, \mathcal{F}_n \subseteq \mathcal{E}_n$ such that $\mathcal{Z} = \prod_{i=1}^n \mathcal{F}_i$ is the smallest subset of $\mathcal{X}$ contains $\mathcal{L}$.

for i from 1 to n **do**

if $\mathcal{F}_i = \{x_i^0\}$ **then**

return $Sq^{k_i}(u_i) := x_i^0$

else

Conclude nothing about $Sq^{k_i}(u_i)$.

There are some cases in which we may fail in doing the comparison, for instance, when "other terms" is trivial, in this case u is primitive, or when Sq^{2^k}(other terms) is trivial. In those cases, we will try to compute the remaining Steenrod operations by using the Evens norm map. However, this method only works for groups of order not exceeding 32.

Examples.

1. Continue the calculation with the Modular group $G = Mod_{16}$ as in the previous example. Let C be greatest central elementary abelian subgroup of G. Then the 2-rank of C is 1, and $\mathrm{H}^*(G, \mathbb{F}_2)$ is generated by an element u_1. The computer calculations give us that $\mu^*(a_1) = a_1 \otimes 1$ and $\mu^*(b_1) = b_1 \otimes 1$. It means that a_1 and b_1 are primitive. We also have

$$\mu^*(c_3) = c_3 \otimes 1 + a_1 \otimes u_1^2$$
$$\mu^*(d_4) = d_4 \otimes 1 + b_1^2 \otimes u_1^2 + u_1^4.$$

Then

$$\begin{aligned} Sq^2(\mu^*(d_4)) &= Sq^2(d_4 \otimes 1 + b_1^2 \otimes u_1^2 + u_1^4) \\ &= Sq^2(d_4) \otimes 1 + b_1^4 \otimes u_1^2 + b_1^2 \otimes u_1^4. \end{aligned} \tag{4.2.8}$$

Moreover, we have

$$\mu^*(d_4 b_1^2) = d_4 b_1^2 \otimes 1 + b_1^4 \otimes u_1^2 + b_1^2 \otimes u_1^4, \tag{4.2.9}$$

and

$$\mu^*(d_4b_1^2 + d_4a_1b_1) = d_4b_1^2 \otimes 1 + d_4a_1b_1 \otimes 1 + b_1^4 \otimes u_1^2 + b_1^2 \otimes u_1^4 + a_1b_1 \otimes u_1^4. \quad (4.2.10)$$

By comparing (4.2.8) with (4.2.9) and (4.2.10), it is easy to see that $d_4b_1^2$ is the only element that satisfies the equation $Sq^2(\mu^*(d_4)) = \mu^*(Sq^2(d_4))$. So we may conclude that $Sq^2(d_4) = d_4b_1^2$.

2. In the case of the small group $G = SmallGroup(32, 8)$, the comodule map method is not enough to determine all Steenrod operations. The cohomology ring $\mathrm{H}^*(G, \mathbb{F}_2)$ is generated by the elements $a_1, b_1, c_2, d_2, e_3, f_5, g_5, h_6, k_8$. After using the detection method, inflation and comodule map, there remain three undetermined actions: $Sq^1(f_5)$, $Sq^1(g_5)$ and $Sq^1(e_3)$. We can see why the comodule map method has no effect on these actions from the following calculation for $Sq^1(f_5)$. Let C be the greatest central elementary abelian subgroup of G. Then C is a cyclic group of order 2, and the cohomology ring $\mathrm{H}^*(C, \mathbb{F}_2)$ is generated by u_1. From our computations, $Sq^1(f_5)$ has two candidates $d_2^2b_1^2 + b_1^3e_3$ and $d_2^2b_1^2$ with their images under comodule map μ^* given by

$$\mu^*(d_2^2b_1^2 + b_1^3e_3) = d_2^2b_1^2 \otimes 1 + b_1^3e_3 \otimes 1,$$
$$\mu^*(d_2^2b_1^2) = d_2^2b_1^2 \otimes 1.$$

However, we have

$$\mu^*(f_5) = f_5 \otimes 1 + b_1^3 \otimes u_1^2 + a_1 \otimes u_1^4,$$

and then

$$Sq^1(\mu^*(f_5)) = Sq^1(f_5) \otimes 1.$$

Therefore, both of the above candidates are suitable, hence it means that we cannot decide which one equals to $Sq^1(f_5)$.

4.2.4 Special cases

The computation of Steenrod operations in the cases of cyclic groups and direct products is simpler and faster.

Cyclic groups:

Let $G = C_{2^r}, r \geq 2$ be a cyclic group of order 2^r. Then $\mathrm{H}^*(G, \mathbb{F}_2) \cong \mathbb{F}_2[u, v]/(u^2)$ with $\deg u = 1, \deg v = 2$. We have the following result with a notice that the Bockstein $\beta = Sq^1$.

Proposition 4.2.7. *$\beta(v) = 0$.*

Proof. From the long exact sequence associated to the short exact sequence of coefficients

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{F}_2 \to 0$$

we take the following segment

$$\ldots \to \mathrm{H}^2(G, \mathbb{Z}) \cong \mathbb{Z}/2^r\mathbb{Z} \xrightarrow{\times 2} \mathrm{H}^2(G, \mathbb{Z}) \cong \mathbb{Z}/2^r\mathbb{Z} \xrightarrow{\pi^*} \mathrm{H}^2(G, \mathbb{F}_2) \cong \mathbb{F}_2 \xrightarrow{\delta} \mathrm{H}^3(G, \mathbb{Z}) \to \ldots,$$

where δ is the connecting homomorphism. It is easy to see that the first map is not surjective, then $\pi^* \neq 0$, and hence π^* is surjective. It follows that δ is trivial. Therefore, $\beta = \pi^* \circ \delta = 0$. $\square$

Direct products:

Let $G = H \times K$ be the direct product of groups H and K. Then we have

$$\mathrm{H}^*(G, \mathbb{F}_p) \cong \mathrm{H}^*(H, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{H}^*(K, \mathbb{F}_p).$$

If the Steenrod operations on $\mathrm{H}^*(H,\mathbb{F}_p)$ and $\mathrm{H}^*(K,\mathbb{F}_p)$ have been computed already, then the Steenrod operations on $\mathrm{H}^*(G,\mathbb{F}_p)$ are determined by the Cartan formula

$$Sq^n(x\otimes y)=\sum_{i+j=n} Sq^i(x)\otimes Sq^j(y). \tag{4.2.11}$$

4.2.5 Steenrod squares from inflation, transfer and the norm

Inflation and transfer map:

Since the action of the Steenrod algebra $\mathcal{A}_p$ commutes with the inflation and the transfer map, we may use these maps to calculate Steenrod operations in some certain cases.

Let G be a finite 2-group. For a generator u of $\mathrm{H}^*(G,\mathbb{F}_2)$, suppose that there is an inflation $\inf^G_{G/H}$ and an element $x\in \mathrm{H}^*(G/H,\mathbb{F}_2)$ such that $\inf^G_{G/H}(x)=u+$ other terms. Then $\inf^G_{G/H}(Sq^{2^r}(x))=Sq^{2^r}(\inf^G_{G/H}(x))=Sq^{2^r}(u)+Sq^{2^r}(\text{other terms})$. If $Sq^{2^r}(x)$ has been computed and $Sq^{2^r}(\text{other terms})$ is determined, then $Sq^{2^r}(u)$ is induced. The similar argument is also applied for the transfer in place of the inflation.

***Remark* 4.2.8.** + Note that we mostly consider the inflations from the maximal quotient groups and the transfers from the maximal subgroups.

+ The inflation is already implemented in the Sage package by Green and King (see [18]), and the transfer is one of our implementations.

Examples. The group $G=\mathit{SmallGroup}(32,2)$ is undetectable, and the detection method doesn't work for this group at all. However, we can use the inflation map to calculate all Steenrod squares on the cohomology of G. The cohomology ring of this group has a minimal generating set $\mathcal{G}=\{a_1,b_1,c_2,d_2,e_2,f_2,g_2\}$. For any generator u in $\mathcal{G}$, there exists an inflation $\inf^G_{G/H}$ and an element $a\in \mathrm{H}^*(G/H,\mathbb{F}_2)$ such that

$u = \inf^G_{G/H}(a)$. For example, for $u = e_2$, there is a inflation from a maximal quotient group isomorphic to $SmallGroup(16, 4)$ and an element a such that $Sq^1(a) = 0$. Then $Sq^1(e_2) = \inf^G_{\_}(Sq^1(a)) = \inf^G_{\_}(0) = 0$. Similarly, there is another inflation and an element $a, Sq^1(a) = 0$, in the domain of this map such that $\inf^G_{\_}(a) = g_2 + e_2$. Hence, $Sq^1(g_2) = \inf^G_{\_}(Sq^1(a)) + Sq^1(e_2) = \inf^G_{\_}(0) = 0$.

The norm map:

By definition, we may determine all Steenrod operations on a given finite group G if we know the norm map $\mathcal{N}orm_G^{G\times C_p}$. The Steenrod operations for all 2-groups of order 32 have been calculated by using the detection method, the inflation, the transfer and the comodule map, except some actions for two groups $SmallGroup(32, 8)$ and $SmallGroup(32, 32)$. By using the norm, we are hopefully able to complete the computation of the Steenrod squares for all groups in this family.

Examples. Consider the small group $G = SmallGroup(32, 32)$. We have $a_1, b_1, c_1, d_3, e_3, f_4, g_4$ are the generators of $\mathrm{H}^*(G, \mathbb{F}_2)$. The action $Sq^2(g_4)$ is still undetermined after using the other methods. Let C_2 be the cyclic group of order 2. The cohomology ring $\mathrm{H}^*(C_2, \mathbb{F}_2)$ is generated by an element u_1. Compute the norm from G to $G \times C_2$, we have

$$
\begin{aligned}
\mathcal{N}orm_G^{G\times C}(g_4) =& g_4^2 \otimes 1 + f_4 a_1^2 b_1 \otimes u_1 \\
&+ g_4 a_1 c_1 \otimes u_1^2 + g_4 a_1^2 \otimes u_1^2 \\
&+ f_4 b_1 c_1 \otimes u_1^2 + f_4 a_1 c_1 \otimes u_1^2 \\
&+ a_1 b_1 d_3 \otimes u_1^3 + a_1^2 e_3 \otimes u_1^3 \\
&+ g_4 \otimes u_1^4.
\end{aligned}
$$

Then we have $Sq^2(g_4) = g_4 a_1 c_1 + g_4 a_1^2 + f_4 b_1 c_1 + f_4 a_1 c_1$.

Unfortunately, the computation of the norm map $\mathcal{N}orm_G^{G\times C_p}$ always takes quite a long time and consumes a big amount of memory. So far, 2-groups of order 32 is the biggest groups on which we perform the calculation of the norm map. The running time for $SmallGroup(32, 32)$ is approximate 2 months.

Chapter 5

Experimental results

In this chapter, we present some selective results from our computations of Steenrod operations on the mod-2 cohomology rings of finite groups. We use the computation of cohomology rings from the Sage package of D. Green and S. King (see [18]) in our implementations, so we adopt their notations in the representation of cohomology rings. In the Green and King's package, a cohomology ring is represented as a polynomial ring modulo a set of relations. The ring generators are denoted by the letters "a, b, c" with two indices, the first indicates the degree and the second is to distinguish generators of the same letter and degree. A generator with nilpotent restriction to the centre of the Sylow subgroup is denoted by letter "b", whilst a *Duflot element*, whose restriction to the centre of the Sylow subgroup is non-nilpotent, is denoted by "c".

For each group, we provide a minimal generating set of the corresponding cohomology ring, and then the action of Steenrod squares Sq^{2^i} on each generator u for $2^i \leq \deg u$. The results are from the first run of our program, and they require more than a hundred pages to present here. Instead, we will give here some of the most interesting results. All the results are available on the World Wide Web at http://users.minet.uni-jena.de/~ti63duv/. Again, the notation $SmallGroup(n, r)$ denotes a group of order n and number r in the small group library of the computer algebra system GAP.

5.1 Steenrod operations for small 2-groups

The Steenrod operations on the cohomology rings of 2-groups of small order were calculated by Rusin [25], and by Guillot [19], but not completely. Rusin calculated for all 2-groups of order 32, except two groups of number 8 and 44. Guillot calculated for 13 of the 14 groups of order 16, 28 of the 51 groups of order 32, and 61 of the 267 groups of order 64.

The first run of our program could be able to compute the Steenrod squares on the cohomology rings of the following 2-groups:

- All groups of order less than or equal 16.
- All groups of order 32 (the actions $Sq^1 a_{5,2}, Sq^1 b_{5,4}$ in the SmallGroup(32,8) are in the process of computing and will be updated soon).
- More than 210 groups of the 267 groups of order 64.
- Several groups of order 128.

5.1.1 The groups of order 32

1. $SmallGroup(32, 1) \cong C_{32}$

   Minimal generating set: $c_{1,0}, c_{2,0}$.

   $Sq^1 c_{2,0} = 0$

2. $SmallGroup(32, 2)$

   Minimal generating set: $a_{1,0}, a_{1,1}, a_{2,0}, a_{2,1}, c_{2,2}, c_{2,3}, c_{2,4}$.

   $Sq^1 a_{2,0} = a_{2,0}a_{1,1} + c_{2,3}a_{1,0} + c_{2,2}a_{1,1}$

   $Sq^1 a_{2,1} = a_{2,0}a_{1,1} + c_{2,4}a_{1,0} + c_{2,3}a_{1,1}$

   $Sq^1 c_{2,2} = 0$

   $Sq^1 c_{2,3} = c_{2,3}a_{1,1} + c_{2,3}a_{1,0}$

   $Sq^1 c_{2,4} = 0$

3. $SmallGroup(32, 3) \cong C_4 \times C_8$

   Minimal generating set: $c_{1,0}, c_{1,1}, c_{2,1}, c_{2,2}$.

   $Sq^1 c_{2,1} = 0$

   $Sq^1 c_{2,2} = 0$

4. $SmallGroup(32, 4)$

   Minimal generating set: $a_{1,0}, a_{1,1}, c_{2,1}, c_{2,2}$.

   $Sq^1 c_{2,1} = 0$

   $Sq^1 c_{2,2} = c_{2,1}a_{1,0}$

5. $SmallGroup(32, 5)$

   Minimal generating set: $a_{1,0}, b_{1,1}, a_{2,1}, c_{2,2}, c_{2,3}$.

   $Sq^1 a_{2,1} = a_{2,1}b_{1,1} + c_{2,2}a_{1,0}$

   $Sq^1 c_{2,2} = c_{2,2}b_{1,1} + c_{2,2}a_{1,0}$

   $Sq^1 c_{2,3} = 0$

6. $SmallGroup(32, 6)$

   Minimal generating set: $a_{1,0}, b_{1,1}, b_{2,1}, b_{2,2}, b_{2,3}, b_{3,4}, b_{3,5}, c_{4,8}$.

$$Sq^1 b_{2,1} = 0$$

$$Sq^1 b_{2,2} = b_{3,4}$$

$$Sq^1 b_{2,3} = b_{2,3}b_{1,1} + b_{2,2}a_{1,0}$$

$$Sq^1 b_{3,4} = 0$$

$$Sq^2 b_{3,4} = b_{2,2}b_{3,5} + b_{2,1}b_{3,5} + b_{2,1}b_{3,4} + c_{4,8}a_{1,0}$$

$$Sq^1 b_{3,5} = b_{2,3}^2 + b_{2,1}b_{2,3}$$

$$Sq^2 b_{3,5} = b_{1,1}^2 b_{3,5} + b_{2,3}b_{3,5} + b_{2,3}^2 b_{1,1} + b_{2,1}b_{2,2}a_{1,0} + c_{4,8}b_{1,1}$$

$$Sq^1 c_{4,8} = b_{2,2}b_{3,5} + b_{2,1}b_{3,4} + b_{2,1}b_{2,2}a_{1,0} + c_{4,8}a_{1,0}$$

$$Sq^2 c_{4,8} = c_{4,8}b_{1,1}^2 + b_{2,3}c_{4,8} + b_{2,2}c_{4,8} + b_{2,1}c_{4,8}$$

7. $SmallGroup(32, 7)$

   Minimal generating set: $a_{1,0}, b_{1,1}, b_{2,1}, b_{2,2}, b_{3,3}, b_{3,4}, b_{4,6}, c_{4,7}$.

$$Sq^1 b_{2,1} = 0$$

$$Sq^1 b_{2,2} = b_{2,2}b_{1,1}$$

$$Sq^1 b_{3,3} = 0$$

$$Sq^2 b_{3,3} = b_{2,2}b_{3,3} + b_{2,1}b_{2,2}b_{1,1} + c_{4,7}a_{1,0}$$

$$Sq^1 b_{3,4} = b_{2,2}^2 + b_{2,1}^2$$

$$Sq^2 b_{3,4} = b_{1,1}^2 b_{3,4} + b_{2,2}b_{3,4} + b_{2,2}^2 b_{1,1} + b_{2,1}b_{3,4} + b_{2,1}^2 b_{1,1} + c_{4,7}b_{1,1}$$

$$Sq^1 b_{4,6} = b_{2,2}b_{3,3} + b_{2,1}b_{3,4} + b_{2,1}^2 b_{1,1}$$

$$Sq^2 b_{4,6} = b_{2,2}b_{4,6} + b_{2,2}^2 b_{1,1}^2 + b_{2,2}^3 + b_{2,1}c_{4,7}$$

$$Sq^1 c_{4,7} = b_{2,2}b_{3,3} + b_{2,1}b_{3,4} + b_{2,1}b_{2,2}b_{1,1} + b_{2,1}^2 b_{1,1}$$

$$Sq^2 c_{4,7} = b_{2,1}b_{4,6} + b_{2,1}b_{2,2}^2 + b_{2,1}^2 b_{2,2} + b_{2,1}^3 + c_{4,7}b_{1,1}^2 + b_{2,2}c_{4,7} + b_{2,1}c_{4,7}$$

8. $SmallGroup(32, 8)$

   Minimal generating set: $a_{1,0}, a_{1,1}, a_{2,1}, b_{2,2}, a_{3,3}, a_{5,2}, b_{5,4}, a_{6,4}, c_{8,6}$.

$Sq^1 a_{2,1} = 0$

$Sq^1 b_{2,2} = b_{2,2}a_{1,1} + a_{1,1}^3$

$Sq^1 a_{3,3} = a_{2,1}a_{1,1}^2$

$Sq^2 a_{3,3} = a_{5,2} + b_{2,2}a_{3,3} + a_{1,1}^2 a_{3,3}$

$Sq^1 a_{5,2} =$?[1]

$Sq^2 a_{5,2} = a_{1,1}^2 b_{5,4}$

$Sq^4 a_{5,2} = c_{8,6}a_{1,0}$

$Sq^1 b_{5,4} =$?[2]

$Sq^2 b_{5,4} = a_{1,1}^2 b_{5,4} + b_{2,2}a_{1,1}^2 a_{3,3}$

$Sq^4 b_{5,4} = b_{2,2}^2 b_{5,4} + b_{2,2}^4 a_{1,1} + c_{8,6}a_{1,1} + c_{8,6}a_{1,0}$

$Sq^1 a_{6,4} = b_{2,2}^2 a_{3,3} + b_{2,2}a_{1,1}^2 a_{3,3}$

$Sq^2 a_{6,4} = b_{2,2}^2 a_{1,1}a_{3,3} + b_{2,2}^3 a_{1,1}^2 + a_{6,4}a_{1,1}^2$

$Sq^4 a_{6,4} = b_{2,2}^2 a_{6,4} + b_{2,2}^3 a_{1,1}a_{3,3} + b_{2,2}a_{6,4}a_{1,1}^2 + a_{2,1}c_{8,6}$

$Sq^1 c_{8,6} = b_{2,2}^2 a_{1,1}^2 a_{3,3}$

$Sq^2 c_{8,6} = b_{2,2}^3 a_{1,1}a_{3,3}$

$Sq^4 c_{8,6} = b_{2,2}^6 + b_{2,2}^5 a_{1,1}^2 + b_{2,2}^2 c_{8,6} + b_{2,2}c_{8,6}a_{1,1}^2$

9. $SmallGroup(32, 9)$

   Minimal generating set: $a_{1,0}, b_{1,1}, b_{2,1}, c_{2,2}, c_{2,3}$.

   $Sq^1 b_{2,1} = b_{2,1}b_{1,1} + c_{2,2}b_{1,1}$

   $Sq^1 c_{2,2} = 0$

   $Sq^1 c_{2,3} = c_{2,3}b_{1,1} + c_{2,3}a_{1,0}$

10. $SmallGroup(32, 10)$

[1] The action will be soon updated
[2] The action will be soon updated

Minimal generating set: $a_{1,0}, a_{1,1}, a_{2,1}, c_{2,2}, a_{3,3}, c_{4,4}$.

$Sq^1a_{2,1} = a_{2,1}a_{1,1} + c_{2,2}a_{1,1}$

$Sq^1c_{2,2} = 0$

$Sq^1a_{3,3} = c_{2,2}a_{1,1}^2$

$Sq^2a_{3,3} = c_{4,4}a_{1,0}$

$Sq^1c_{4,4} = 0$

$Sq^2c_{4,4} = 0$

11. $SmallGroup(32, 11)$

    Minimal generating set: $a_{1,0}, b_{1,1}, a_{2,1}, b_{2,2}, a_{3,3}, c_{4,4}$.

    $Sq^1a_{2,1} = a_{2,1}b_{1,1}$

    $Sq^1b_{2,2} = 0$

    $Sq^1a_{3,3} = 0$

    $Sq^2a_{3,3} = b_{2,2}a_{3,3} + c_{4,4}a_{1,0}$

    $Sq^1c_{4,4} = 0$

    $Sq^2c_{4,4} = c_{4,4}b_{1,1}^2 + b_{2,2}c_{4,4}$

12. $SmallGroup(32, 12)$

    Minimal generating set: $a_{1,0}, a_{1,1}, c_{2,1}, c_{2,2}$.

    $Sq^1c_{2,1} = c_{2,1}a_{1,0}$

    $Sq^1c_{2,2} = 0$

13. $SmallGroup(32, 13)$

    Minimal generating set: $a_{1,0}, a_{1,1}, c_{2,1}, c_{2,2}$.

    $Sq^1c_{2,1} = 0$

    $Sq^1c_{2,2} = c_{2,2}a_{1,0} + c_{2,1}a_{1,1}$

14. $SmallGroup(32, 14)$

    Minimal generating set: $a_{1,0}, a_{1,1}, c_{2,1}, c_{2,2}$.

    $Sq^1 c_{2,1} = 0$

    $Sq^1 c_{2,2} = c_{2,2} a_{1,0}$

15. $SmallGroup(32, 15)$

    Minimal generating set: $a_{1,0}, a_{1,1}, b_{2,1}, b_{3,1}, c_{4,2}$.

    $Sq^1 b_{2,1} = 0$

    $Sq^1 b_{3,1} = b_{2,1}^2$

    $Sq^2 b_{3,1} = b_{2,1} b_{3,1} + c_{4,2} a_{1,0}$

    $Sq^1 c_{4,2} = 0$

    $Sq^2 c_{4,2} = b_{2,1}^3 + b_{2,1} c_{4,2} + c_{4,2} a_{1,0} a_{1,1}$

16. $SmallGroup(32, 16)$

    Minimal generating set: $c_{1,0}, c_{1,1}, c_{2,2}$.

    $Sq^1 c_{2,2} = 0$

17. $SmallGroup(32, 17) \cong Mod32$

    Minimal generating set: $a_{1,0}, b_{1,1}, a_{3,1}, c_{4,2}$.

    $Sq^1 a_{3,1} = 0$

    $Sq^2 a_{3,1} = b_{1,1}^2 a_{3,1} + c_{4,2} a_{1,0}$

    $Sq^1 c_{4,2} = b_{1,1}^2 a_{3,1}$

    $Sq^2 c_{4,2} = c_{4,2} b_{1,1}^2$

18. $SmallGroup(32, 18) \cong D_{32}$

    Minimal generating set: $b_{1,0}, b_{1,1}, c_{2,2}$.

$Sq^1 c_{2,2} = c_{2,2} b_{1,1} + c_{2,2} b_{1,0}$

19. $SmallGroup(32, 19) \cong SD_{32}$

    Minimal generating set: $a_{1,0}, b_{1,1}, b_{3,1}, c_{4,2}$.

    $Sq^1 b_{3,1} = 0$

    $Sq^2 b_{3,1} = b_{1,1}^2 b_{3,1} + c_{4,2} b_{1,1}$

    $Sq^1 c_{4,2} = 0$

    $Sq^2 c_{4,2} = c_{4,2} b_{1,1}^2$

20. $SmallGroup(32, 20) \cong Q_{32}$

    Minimal generating set: $a_{1,0}, a_{1,1}, c_{4,0}$.

    $Sq^1 c_{4,0} = 0$

    $Sq^2 c_{4,0} = 0$

21. $SmallGroup(32, 21)$

    Minimal generating set: $c_{1,0}, c_{1,1}, c_{1,2}, c_{2,4}, c_{2,5}$.

    $Sq^1 c_{2,4} = 0$

    $Sq^1 c_{2,5} = 0$

22. $SmallGroup(32, 22) \cong SmallGroup(16, 3) \times C_2$

    Minimal generating set: $a_{1,0}, b_{1,1}, c_{1,2}, b_{2,4}, c_{2,5}, c_{2,6}$.

    $Sq^1 b_{2,4} = b_{2,4} b_{1,1} + c_{2,5} b_{1,1} + c_{2,6} a_{1,0}$

    $Sq^1 c_{2,5} = 0$

    $Sq^1 c_{2,6} = c_{2,6} b_{1,1} + c_{2,6} a_{1,0}$

23. $SmallGroup(32, 23) \cong SmallGroup(16, 4) \times C_2$

    Minimal generating set: $a_{1,0}, a_{1,1}, c_{1,2}, c_{2,4}, c_{2,5}$.

$Sq^1 c_{2,4} = 0$

$Sq^1 c_{2,5} = c_{2,5} a_{1,0}$

24. $SmallGroup(32, 24)$

    Minimal generating set: $a_{1,0}, a_{1,2}, b_{1,1}, c_{2,4}, b_{3,6}, c_{4,9}$.

    $Sq^1 c_{2,4} = 0$

    $Sq^1 b_{3,6} = c_{2,4} a_{1,2}^2$

    $Sq^2 b_{3,6} = b_{1,1}^2 b_{3,6} + c_{2,4} b_{1,1}^3 + c_{4,9} a_{1,0} + c_{2,4}^2 b_{1,1}$

    $Sq^1 c_{4,9} = 0$

    $Sq^2 c_{4,9} = c_{4,9} b_{1,1}^2 + c_{4,9} a_{1,2}^2$

25. $SmallGroup(32, 25) \cong D_8 \times C_4$

    Minimal generating set: $a_{1,0}, b_{1,1}, b_{1,2}, c_{2,4}, c_{2,5}$.

    $Sq^1 c_{2,4} = c_{2,4} b_{1,2} + c_{2,4} b_{1,1} + c_{2,4} a_{1,0}$

    $Sq^1 c_{2,5} = 0$

26. $SmallGroup(32, 26) \cong Q_8 \times C_4$

    Minimal generating set: $a_{1,0}, a_{1,1}, a_{1,2}, c_{2,4}, c_{4,6}$.

    $Sq^1 c_{2,4} = 0$

    $Sq^1 c_{4,6} = 0$

    $Sq^2 c_{4,6} = 0$

27. $SmallGroup(32, 27)$

    Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, b_{2,4}, c_{2,5}, c_{2,6}$.

    $Sq^1 b_{2,4} = b_{2,4} b_{1,2} + b_{2,4} b_{1,1} + c_{2,6} b_{1,1} + c_{2,5} b_{1,2}$

    $Sq^1 c_{2,5} = c_{2,5} b_{1,1} + c_{2,5} b_{1,0}$

$Sq^1 c_{2,6} = c_{2,6}b_{1,2} + c_{2,6}b_{1,0}$

28. $SmallGroup(32, 28)$

    Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, c_{2,4}, c_{2,5}$.

    $Sq^1 c_{2,4} = c_{2,4}b_{1,2} + c_{2,4}b_{1,0}$

    $Sq^1 c_{2,5} = b_{1,0}^2 b_{1,1} + c_{2,5}b_{1,0}$

29. $SmallGroup(32, 29)$

    Minimal generating set: $a_{1,0}, a_{1,1}, b_{1,2}, c_{2,4}, b_{3,6}, c_{4,9}$.

    $Sq^1 c_{2,4} = c_{2,4}b_{1,2} + c_{2,4}a_{1,0}$

    $Sq^1 b_{3,6} = b_{1,2}b_{3,6} + c_{2,4}a_{1,0}^2$

    $Sq^2 b_{3,6} = c_{4,9}b_{1,2}$

    $Sq^1 c_{4,9} = 0$

    $Sq^2 c_{4,9} = 0$

30. $SmallGroup(32, 30)$

    Minimal generating set: $a_{1,2}, b_{1,0}, b_{1,1}, c_{2,4}, b_{3,6}, b_{3,7}, c_{4,11}$.

    $Sq^1 c_{2,4} = c_{2,4}b_{1,0} + c_{2,4}a_{1,2}$

    $Sq^1 b_{3,6} = a_{1,2}b_{3,7} + c_{2,4}b_{1,1}^2 + c_{2,4}a_{1,2}^2$

    $Sq^2 b_{3,6} = b_{1,0}^2 b_{3,6} + c_{4,11}b_{1,0} + c_{2,4}^2 b_{1,1}$

    $Sq^1 b_{3,7} = a_{1,2}b_{3,7} + c_{2,4}b_{1,1}^2 + c_{2,4}a_{1,2}^2$

    $Sq^2 b_{3,7} = b_{1,1}^2 b_{3,7} + c_{4,11}a_{1,2} + c_{2,4}a_{1,2}b_{1,1}^2$

    $Sq^1 c_{4,11} = c_{2,4}a_{1,2}b_{1,1}^2$

    $Sq^2 c_{4,11} = c_{4,11}b_{1,1}^2 + c_{4,11}b_{1,0}^2 + c_{4,11}a_{1,2}^2 + c_{2,4}^2 b_{1,1}^2 + c_{2,4}^2 a_{1,2}^2$

31. $SmallGroup(32, 31)$

Minimal generating set: $a_{1,1}, b_{1,0}, b_{1,2}, c_{2,4}, b_{3,6}, c_{4,9}$.

$Sq^1 c_{2,4} = c_{2,4} b_{1,0}$

$Sq^1 b_{3,6} = b_{1,0}^3 b_{1,2} + c_{2,4} a_{1,1}^2$

$Sq^2 b_{3,6} = b_{1,0} b_{1,2} b_{3,6} + b_{1,0}^2 b_{3,6} + c_{4,9} b_{1,0} + c_{2,4} b_{1,0}^2 b_{1,2} + c_{2,4} b_{1,0}^3 + c_{4,9} a_{1,1} + c_{2,4}^2 b_{1,0}$

$Sq^1 c_{4,9} = b_{1,0} b_{1,2} b_{3,6} + b_{1,0}^4 b_{1,2} + c_{2,4} b_{1,0}^2 b_{1,2} + c_{2,4} b_{1,0}^3 + c_{2,4} a_{1,1}^2 b_{1,2}$

$Sq^2 c_{4,9} = b_{1,0}^2 b_{1,2} b_{3,6} + b_{1,0}^5 b_{1,2} + c_{4,9} b_{1,0} b_{1,2} + c_{4,9} b_{1,0}^2 + c_{4,9} a_{1,1} b_{1,2} + c_{4,9} a_{1,1}^2 + c_{2,4}^2 b_{1,0}^2$

32. $SmallGroup(32, 32)$

Minimal generating set: $a_{1,0}, a_{1,1}, a_{1,2}, a_{3,2}, a_{3,3}, c_{4,4}, c_{4,5}$.

$Sq^1 a_{3,2} = 0$

$Sq^2 a_{3,2} = a_{1,0} a_{1,1} a_{3,2} + a_{1,0}^2 a_{3,3} + c_{4,5} a_{1,0} + c_{4,4} a_{1,2} + c_{4,4} a_{1,1}$

$Sq^1 a_{3,3} = a_{1,0} a_{3,2}$

$Sq^2 a_{3,3} = a_{1,0} a_{1,1} a_{3,3} + a_{1,0}^2 a_{3,3} + c_{4,5} a_{1,2} + c_{4,5} a_{1,1} + c_{4,5} a_{1,0} + c_{4,4} a_{1,0}$

$Sq^1 c_{4,4} = 0$

$Sq^2 c_{4,4} = 0$

$Sq^1 c_{4,5} = a_{1,0} a_{1,1} a_{3,2} + a_{1,0}^2 a_{3,3}$

$Sq^2 c_{4,5} = c_{4,5} a_{1,0} a_{1,2} + c_{4,5} a_{1,0}^2 + c_{4,4} a_{1,1} a_{1,2} + c_{4,4} a_{1,0} a_{1,2}$

33. $SmallGroup(32, 33)$

Minimal generating set: $a_{1,1}, a_{1,2}, b_{1,0}, a_{3,3}, b_{3,2}, b_{3,4}, c_{4,6}, c_{4,7}$.

$Sq^1 b_{3,2} = a_{1,1} b_{3,2}$

$Sq^2 b_{3,2} = b_{1,0}^2 b_{3,2} + a_{1,1} a_{1,2} a_{3,3} + c_{4,7} b_{1,0} + c_{4,6} a_{1,1}$

$Sq^1 a_{3,3} = a_{1,1} b_{3,2}$

$Sq^2 a_{3,3} = a_{1,1} a_{1,2} a_{3,3} + c_{4,7} a_{1,1} + c_{4,6} a_{1,2}$

$Sq^1 b_{3,4} = a_{1,1} b_{3,2}$

$$Sq^2 b_{3,4} = b_{1,0}^2 b_{3,4} + c_{4,7} b_{1,0} + c_{4,6} b_{1,0} + c_{4,7} a_{1,2}$$

$$Sq^1 c_{4,6} = a_{1,1} a_{1,2} a_{3,3}$$

$$Sq^2 c_{4,6} = c_{4,6} b_{1,0}^2 + c_{4,7} a_{1,1} a_{1,2} + c_{4,6} a_{1,1}^2$$

$$Sq^1 c_{4,7} = 0$$

$$Sq^2 c_{4,7} = c_{4,7} b_{1,0}^2 + c_{4,7} a_{1,2}^2 + c_{4,6} a_{1,1}^2$$

34. $SmallGroup(32, 34)$

    Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, c_{2,4}, c_{2,5}$.

    $$Sq^1 c_{2,4} = b_{1,0}^2 b_{1,1} + c_{2,4} b_{1,0}$$

    $$Sq^1 c_{2,5} = b_{1,0}^2 b_{1,2} + c_{2,5} b_{1,0}$$

35. $SmallGroup(32, 35)$

    Minimal generating set: $a_{1,0}, a_{1,1}, a_{1,2}, c_{2,4}, c_{4,6}$.

    $$Sq^1 c_{2,4} = a_{1,0}^2 a_{1,2} + c_{2,4} a_{1,0}$$

    $$Sq^1 c_{4,6} = 0$$

    $$Sq^2 c_{4,6} = 0$$

36. $SmallGroup(32, 36)$

    Minimal generating set: $c_{1,0}, c_{1,1}, c_{1,2}, c_{2,5}$.

    $$Sq^1 c_{2,5} = 0$$

37. $SmallGroup(32, 37) \cong Mod16 \times C_2$

    Minimal generating set: $a_{1,0}, b_{1,1}, c_{1,2}, a_{3,6}, c_{4,9}$.

    $$Sq^1 a_{3,6} = 0$$

    $$Sq^2 a_{3,6} = b_{1,1}^2 a_{3,6} + c_{4,9} a_{1,0}$$

$Sq^1 c_{4,9} = b_{1,1}^2 a_{3,6}$

$Sq^2 c_{4,9} = c_{4,9} b_{1,1}^2$

38. $SmallGroup(32, 38)$

    Minimal generating set: $a_{1,0}, b_{1,1}, b_{1,2}, c_{4,6}$.

    $Sq^1 c_{4,6} = 0$

    $Sq^2 c_{4,6} = b_{1,1}^5 b_{1,2} + c_{4,6} b_{1,2}^2 + c_{4,6} b_{1,1} b_{1,2} + c_{4,6} b_{1,1}^2$

39. $SmallGroup(32, 39) \cong D_{16} \times C_2$

    Minimal generating set: $b_{1,0}, b_{1,1}, c_{1,2}, c_{2,5}$.

    $Sq^1 c_{2,5} = c_{2,5} b_{1,1} + c_{2,5} b_{1,0}$

40. $SmallGroup(32, 40) \cong SD_{16} \times C_2$

    Minimal generating set: $a_{1,0}, b_{1,1}, c_{1,2}, b_{3,6}, c_{4,9}$.

    $Sq^1 b_{3,6} = 0$

    $Sq^2 b_{3,6} = b_{1,1}^2 b_{3,6} + c_{4,9} b_{1,1}$

    $Sq^1 c_{4,9} = 0$

    $Sq^2 c_{4,9} = c_{4,9} b_{1,1}^2$

41. $SmallGroup(32, 41) \cong Q_{16} \times C_2$

    Minimal generating set: $a_{1,0}, a_{1,1}, c_{1,2}, c_{4,6}$.

    $Sq^1 c_{4,6} = 0$

    $Sq^2 c_{4,6} = 0$

42. $SmallGroup(32, 42)$

    Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, c_{4,6}$.

$Sq^1 c_{4,6} = 0$

$Sq^2 c_{4,6} = c_{4,6} b_{1,2}^2 + c_{4,6} b_{1,1}^2 + c_{4,6} b_{1,0}^2$

43. $SmallGroup(32, 43)$

Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, b_{3,6}, c_{4,9}$.

$Sq^1 b_{3,6} = 0$

$Sq^2 b_{3,6} = b_{1,2}^2 b_{3,6} + b_{1,1}^2 b_{3,6} + c_{4,9} b_{1,1}$

$Sq^1 c_{4,9} = b_{1,2}^2 b_{3,6}$

$Sq^2 c_{4,9} = b_{1,1} b_{1,2}^2 b_{3,6} + c_{4,9} b_{1,2}^2 + c_{4,9} b_{1,1}^2 + c_{4,9} b_{1,0}^2$

44. $SmallGroup(32, 44)$

Minimal generating set: $a_{1,1}, b_{1,0}, b_{1,2}, a_{5,6}, b_{5,5}, c_{8,10}$.

$Sq^1 b_{5,5} = b_{1,0} b_{5,5}$

$Sq^2 b_{5,5} = b_{1,0}^2 b_{5,5} + a_{1,1}^2 a_{5,6}$

$Sq^4 b_{5,5} = c_{8,10} b_{1,0}$

$Sq^1 a_{5,6} = a_{1,1} a_{5,6} + a_{1,1}^2 b_{1,2}^4$

$Sq^2 a_{5,6} = 0$

$Sq^4 a_{5,6} = b_{1,2}^4 a_{5,6} + a_{1,1} b_{1,2}^3 a_{5,6} + c_{8,10} a_{1,1}$

$Sq^1 c_{8,10} = b_{1,2}^4 a_{5,6} + a_{1,1} b_{1,2}^3 a_{5,6} + a_{1,1}^2 b_{1,2}^7$

$Sq^2 c_{8,10} = b_{1,2}^5 a_{5,6} + a_{1,1} b_{1,2}^4 a_{5,6} + a_{1,1}^2 b_{1,2}^8$

$Sq^4 c_{8,10} = a_{1,1}^2 b_{1,2}^{10} + c_{8,10} b_{1,2}^4 + c_{8,10} b_{1,0}^4 + c_{8,10} a_{1,1} b_{1,2}^3 + c_{8,10} a_{1,1}^3 b_{1,2}$

45. $SmallGroup(32, 45)$

Minimal generating set: $c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{2,9}$.

$Sq^1 c_{2,9} = 0$

46. $SmallGroup(32,46) \cong D_8 \times V_4$

    Minimal generating set: $b_{1,0}, b_{1,1}, c_{1,2}, c_{1,3}, c_{2,9}$.

    $Sq^1 c_{2,9} = c_{2,9}b_{1,1} + c_{2,9}b_{1,0}$

47. $SmallGroup(32,47) \cong Q_8 \times V_4$

    Minimal generating set: $a_{1,0}, a_{1,1}, c_{1,2}, c_{1,3}, c_{4,21}$.

    $Sq^1 c_{4,21} = 0$

    $Sq^2 c_{4,21} = 0$

48. $SmallGroup(32,48) \cong SmallGroup(16,13) \times C_2$

    Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, c_{1,3}, c_{4,21}$.

    $Sq^1 c_{4,21} = 0$

    $Sq^2 c_{4,21} = b_{1,0}^5 b_{1,1} + c_{4,21}b_{1,1}^2 + c_{4,21}b_{1,0}b_{1,1} + c_{4,21}b_{1,0}^2$

49. $SmallGroup(32,49) \cong E_{32^+}$

    Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, b_{1,3}, c_{4,21}$.

    $Sq^1 c_{4,21} = 0$

    $Sq^2 c_{4,21} = c_{4,21}b_{1,3}^2 + c_{4,21}b_{1,2}b_{1,3} + c_{4,21}b_{1,2}^2 + c_{4,21}b_{1,1}b_{1,3} + c_{4,21}b_{1,1}^2 + c_{4,21}b_{1,0}b_{1,2} + c_{4,21}b_{1,0}b_{1,1} + c_{4,21}b_{1,0}^2$

50. $SmallGroup(32,50) \cong E_{32^-}$

    Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, b_{1,3}, c_{8,30}$.

    $Sq^1 c_{8,30} = b_{1,0}^3 b_{1,1}^2 b_{1,2} b_{1,3}^3 + b_{1,0}^4 b_{1,1} b_{1,2} b_{1,3}^3 + b_{1,0}^4 b_{1,1}^2 b_{1,3}^3 + b_{1,0}^4 b_{1,1}^2 b_{1,2} b_{1,3}^2 + b_{1,0}^5 b_{1,1} b_{1,3}^3 + b_{1,0}^5 b_{1,1} b_{1,2} b_{1,3}^2 + b_{1,0}^7 b_{1,3}^2 + b_{1,0}^8 b_{1,3}$

    $Sq^2 c_{8,30} = b_{1,0}^5 b_{1,1}^2 b_{1,2} b_{1,3}^2 + b_{1,0}^6 b_{1,2} b_{1,3}^3 + b_{1,0}^6 b_{1,1}^2 b_{1,3}^2 + b_{1,0}^6 b_{1,1}^2 b_{1,2} b_{1,3} + b_{1,0}^7 b_{1,1} b_{1,3}^2 + b_{1,0}^8 b_{1,3}^2 + b_{1,0}^8 b_{1,2} b_{1,3} + b_{1,0}^9 b_{1,3}$

$Sq^4 c_{8,30} = b_{1,0}^7 b_{1,1}^2 b_{1,3}^3 + b_{1,0}^9 b_{1,2} b_{1,3}^2 + b_{1,0}^9 b_{1,1} b_{1,3}^2 + b_{1,0}^9 b_{1,1} b_{1,2} b_{1,3} + b_{1,0}^{10} b_{1,3}^2 + b_{1,0}^{10} b_{1,2} b_{1,3} + b_{1,0}^{10} b_{1,1} b_{1,2} + b_{1,0}^{11} b_{1,3} + b_{1,0}^{11} b_{1,1} + c_{8,30} b_{1,3}^4 + c_{8,30} b_{1,0} b_{1,3}^3 + c_{8,30} b_{1,0}^4$

51. $SmallGroup(32, 51) \cong (C_2)^5$

    Minimal generating set: $c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}, c_{1,4}$.

5.1.2 Groups of order 64

1. $SmallGroup(64, 134) \cong Syl_2(M_{12})$

   Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, b_{2,4}, b_{2,5}, b_{3,9}, c_{4,14}$.

   $Sq^1 b_{2,4} = b_{2,4} b_{1,1}$

   $Sq^1 b_{2,5} = b_{2,5} b_{1,2} + b_{2,5} b_{1,0}$

   $Sq^1 b_{3,9} = 0$

   $Sq^2 b_{3,9} = b_{2,5} b_{3,9} + b_{2,4} b_{2,5} b_{1,2} + c_{4,14} b_{1,0}$

   $Sq^1 c_{4,14} = b_{1,0}^2 b_{3,9} + b_{2,4} b_{1,1}^2 b_{1,2} + b_{2,4}^2 b_{1,1}$

   $Sq^2 c_{4,14} = b_{1,0}^3 b_{3,9} + b_{2,4}^2 b_{2,5} + b_{2,4}^3 + c_{4,14} b_{1,2}^2 + c_{4,14} b_{1,1} b_{1,2} + c_{4,14} b_{1,1}^2 + b_{2,5} c_{4,14}$

2. $SmallGroup(64, 138) \cong Syl_2(A_8) \cong Syl_2(A_9)$

   Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, b_{2,4}, b_{2,5}, b_{2,6}, b_{3,11}, c_{4,18}$.

   $Sq^1 b_{2,4} = b_{2,4} b_{1,1} + b_{2,4} b_{1,0}$

   $Sq^1 b_{2,5} = b_{3,11} + b_{2,5} b_{1,0} + b_{2,4} b_{1,2}$

   $Sq^1 b_{2,6} = b_{2,6} b_{1,2} + b_{2,6} b_{1,0}$

   $Sq^1 b_{3,11} = b_{2,5}^2 + b_{2,4} b_{2,6}$

   $Sq^2 b_{3,11} = b_{2,6} b_{3,11} + b_{2,5} b_{3,11} + b_{2,4} b_{3,11} + c_{4,18} b_{1,0}$

   $Sq^1 c_{4,18} = c_{4,18} b_{1,0}$

   $Sq^2 c_{4,18} = c_{4,18} b_{1,2}^2 + c_{4,18} b_{1,1} b_{1,2} + c_{4,18} b_{1,1}^2 + b_{2,6} c_{4,18} + b_{2,5} c_{4,18} + b_{2,4} c_{4,18}$

3. $SmallGroup(64, 242) \cong Syl_2(M_{21})$

Minimal generating set: $b_{1,0}, b_{1,1}, b_{1,2}, b_{1,3}, b_{3,10}, b_{3,11}, c_{4,18}, c_{4,19}, b_{6,47}$.

$Sq^1 b_{3,10} = 0$

$Sq^2 b_{3,10} = b_{1,0}b_{1,2}b_{3,11} + b_{1,0}b_{1,1}b_{3,10} + b_{1,0}^2 b_{3,10} + b_{1,0}^3 b_{1,1}^2 + c_{4,19}b_{1,2} + c_{4,18}b_{1,3} + c_{4,18}b_{1,2} + c_{4,18}b_{1,0}$

$Sq^1 b_{3,11} = b_{1,0}^2 b_{1,1}^2 + b_{1,0}^4$

$Sq^2 b_{3,11} = b_{1,3}^2 b_{3,10} + b_{1,2}b_{1,3}b_{3,10} + b_{1,2}^2 b_{3,10} + b_{1,1}^2 b_{3,11} + b_{1,1}^2 b_{3,10} + b_{1,0}b_{1,1}b_{3,10} + b_{1,0}b_{1,1}^4 + b_{1,0}^2 b_{3,11} + c_{4,19}b_{1,3} + c_{4,19}b_{1,2} + c_{4,18}b_{1,3} + c_{4,18}b_{1,1}$

$Sq^1 c_{4,18} = b_{1,3}^2 b_{3,10} + b_{1,2}^2 b_{3,11} + b_{1,0}^2 b_{3,11} + b_{1,0}^3 b_{1,1}^2 + b_{1,0}^4 b_{1,1}$

$Sq^2 c_{4,18} = b_{1,3}^3 b_{3,10} + b_{1,2}b_{1,3}^2 b_{3,11} + b_{1,2}^2 b_{1,3}b_{3,11} + b_{1,2}^3 b_{3,11} + b_{1,2}^3 b_{3,10} + b_{1,0}b_{1,1}^2 b_{3,11} + b_{1,0}^2 b_{1,1}b_{3,10} + b_{1,0}^3 b_{1,1}^3 + b_{1,0}^4 b_{1,1}^2 + c_{4,19}b_{1,2}^2 + c_{4,19}b_{1,0}b_{1,3} + c_{4,19}b_{1,0}b_{1,2} + c_{4,18}b_{1,2}b_{1,3} + c_{4,18}b_{1,2}^2 + c_{4,18}b_{1,0}b_{1,3} + c_{4,18}b_{1,0}b_{1,2} + c_{4,18}b_{1,0}b_{1,1}$

$Sq^1 c_{4,19} = b_{1,3}^2 b_{3,11} + b_{1,2}^2 b_{3,11} + b_{1,2}^2 b_{3,10} + b_{1,1}^2 b_{3,11} + b_{1,0}b_{1,1}^4 + b_{1,0}^2 b_{3,11} + b_{1,0}^2 b_{1,1}^3 + b_{1,0}^4 b_{1,1} + b_{1,0}^5$

$Sq^2 c_{4,19} = b_{1,3}^3 b_{3,11} + b_{1,3}^3 b_{3,10} + b_{1,2}b_{1,3}^2 b_{3,10} + b_{1,2}^2 b_{1,3}b_{3,10} + b_{1,2}^3 b_{3,11} + b_{1,1}^3 b_{3,11} + b_{1,1}^3 b_{3,10} + b_{1,0}b_{1,1}^2 b_{3,11} + b_{1,0}b_{1,1}^2 b_{3,10} + b_{1,0}^2 b_{1,1}^4 + b_{1,0}^3 b_{3,11} + b_{1,0}^3 b_{1,1}^3 + b_{1,0}^6 + c_{4,19}b_{1,2}b_{1,3} + c_{4,19}b_{1,2}^2 + c_{4,19}b_{1,1}^2 + c_{4,19}b_{1,0}b_{1,2} + c_{4,19}b_{1,0}b_{1,1} + c_{4,19}b_{1,0}^2 + c_{4,18}b_{1,3}^2 + c_{4,18}b_{1,2}^2 + c_{4,18}b_{1,1}^2 + c_{4,18}b_{1,0}b_{1,1}$

$Sq^1 b_{6,47} = b_{1,3}^4 b_{3,11} + b_{1,2}b_{1,3}^3 b_{3,10} + b_{1,2}^2 b_{1,3}^2 b_{3,11} + b_{1,2}^2 b_{1,3}^2 b_{3,10} + b_{1,2}^3 b_{1,3}b_{3,11} + b_{1,2}^4 b_{3,10} + b_{1,1}^4 b_{3,11} + b_{1,0}b_{1,1}^3 b_{3,11} + b_{1,0}b_{1,1}^6 + b_{1,0}^2 b_{1,1}^2 b_{3,11} + b_{1,0}^2 b_{1,1}^5 + b_{1,0}^3 b_{1,1}b_{3,10} + b_{1,0}^4 b_{3,11} + b_{1,0}^4 b_{1,1}^3 + b_{1,0}^5 b_{1,1}^2 + b_{1,0}^6 b_{1,1} + b_{6,47}b_{1,1} + c_{4,19}b_{1,1}^3 + c_{4,19}b_{1,0}^2 b_{1,1} + c_{4,18}b_{1,2}b_{1,3}^2 + c_{4,18}b_{1,2}^2 b_{1,3} + c_{4,18}b_{1,0}b_{1,1}^2 + c_{4,18}b_{1,0}^2 b_{1,3} + c_{4,18}b_{1,0}^3$

$Sq^2 b_{6,47} = b_{1,3}^5 b_{3,11} + b_{1,2}b_{1,3}^4 b_{3,10} + b_{1,2}^3 b_{1,3}^2 b_{3,11} + b_{1,2}^3 b_{1,3}^2 b_{3,10} + b_{1,2}^4 b_{1,3}b_{3,11} + b_{1,1}^5 b_{3,11} + b_{1,2}^4 b_{1,3}b_{3,10} + b_{1,0}^2 b_{1,1}^3 b_{3,11} + b_{1,0}^3 b_{1,1}^2 b_{3,11} + b_{1,0}^3 b_{1,1}^2 b_{3,10} + b_{1,0}^3 b_{1,1}^5 + b_{1,0}^4 b_{1,1}b_{3,11} + b_{1,0}^4 b_{1,1}^4 +$

$b_{1,0}^5b_{3,10}+b_{1,0}^6b_{1,1}^2+b_{1,0}^7b_{1,1}+b_{1,0}^8+b_{6,47}b_{1,1}^2+b_{6,47}b_{1,0}b_{1,1}+c_{4,19}b_{1,3}b_{3,10}+c_{4,19}b_{1,3}^4+c_{4,19}b_{1,2}b_{3,11}+c_{4,19}b_{1,2}b_{3,10}+c_{4,19}b_{1,2}^3b_{1,3}+c_{4,19}b_{1,1}b_{3,10}+c_{4,19}b_{1,0}b_{3,11}+c_{4,19}b_{1,0}b_{1,1}^3+c_{4,19}b_{1,0}^4+c_{4,18}b_{1,3}b_{3,11}+c_{4,18}b_{1,3}b_{3,10}+c_{4,18}b_{1,2}b_{3,11}+c_{4,18}b_{1,2}b_{1,3}^3+c_{4,18}b_{1,1}b_{3,11}+c_{4,18}b_{1,1}b_{3,10}+c_{4,18}b_{1,1}^4+c_{4,18}b_{1,0}b_{3,10}$

$Sq^4b_{6,47}=b_{1,3}^7b_{3,11}+b_{1,2}b_{1,3}^6b_{3,11}+b_{1,2}^4b_{1,3}^3b_{3,11}+b_{1,2}^5b_{1,3}^2b_{3,11}+b_{1,2}^5b_{1,3}^2b_{3,10}+b_{1,2}^7b_{3,10}+b_{1,2}^6b_{1,3}b_{3,11}+b_{1,2}^6b_{1,3}b_{3,10}+b_{1,1}^7b_{3,11}+b_{1,0}^2b_{1,1}^5b_{3,11}+b_{1,0}^2b_{1,1}^8+b_{1,0}^3b_{1,1}^7+b_{1,0}^4b_{1,1}^6+b_{1,0}^5b_{1,1}^2b_{3,11}+b_{1,0}^6b_{1,1}b_{3,11}+b_{1,0}^7b_{3,11}+b_{1,0}^7b_{1,1}^3+b_{1,0}^8b_{1,1}^2+b_{6,47}b_{1,3}b_{3,11}+b_{6,47}b_{1,2}b_{3,10}+b_{6,47}b_{1,2}^3b_{1,3}+b_{6,47}b_{1,1}b_{3,11}+b_{6,47}b_{1,1}^4+b_{6,47}b_{1,0}b_{3,11}+b_{6,47}b_{1,0}b_{3,10}+b_{6,47}b_{1,0}b_{1,1}^3+b_{6,47}b_{1,0}^3b_{1,1}+b_{6,47}b_{1,0}^4+c_{4,19}b_{1,3}^3b_{3,11}+c_{4,19}b_{1,3}^3b_{3,10}+c_{4,19}b_{1,2}^2b_{1,3}^4+c_{4,19}b_{1,2}^3b_{3,11}+c_{4,19}b_{1,2}^3b_{3,10}+c_{4,19}b_{1,2}^3b_{1,3}^3+c_{4,19}b_{1,2}^4b_{1,3}^2+c_{4,19}b_{1,2}^5b_{1,3}+c_{4,19}b_{1,1}^3b_{3,11}+c_{4,19}b_{1,1}^3b_{3,10}+c_{4,19}b_{1,0}b_{1,1}^2b_{3,11}+c_{4,19}b_{1,0}b_{1,1}^2b_{3,10}+c_{4,19}b_{1,0}b_{1,1}^5+c_{4,19}b_{1,0}^2b_{1,1}b_{3,11}+c_{4,19}b_{1,0}^2b_{1,1}b_{3,10}+c_{4,19}b_{1,0}^4b_{1,1}^2+c_{4,18}b_{1,3}^3b_{3,11}+c_{4,18}b_{1,2}b_{1,3}^2b_{3,10}+c_{4,18}b_{1,2}^2b_{1,3}b_{3,11}+c_{4,18}b_{1,2}^2b_{1,3}b_{3,10}+c_{4,18}b_{1,2}^2b_{1,3}^4+c_{4,18}b_{1,2}^3b_{3,11}+c_{4,18}b_{1,2}^4b_{1,3}^2+c_{4,18}b_{1,2}^5b_{1,3}+c_{4,18}b_{1,1}^3b_{3,11}+c_{4,18}b_{1,0}b_{1,1}^2b_{3,11}+c_{4,18}b_{1,0}b_{1,1}^2b_{3,10}+c_{4,18}b_{1,0}b_{1,1}^5+c_{4,18}b_{1,0}^2b_{1,1}b_{3,10}+c_{4,18}b_{1,0}^3b_{3,11}+c_{4,18}b_{1,0}^6+c_{4,19}^2b_{1,3}^2+c_{4,19}^2b_{1,2}b_{1,3}+c_{4,19}^2b_{1,2}^2+c_{4,19}^2b_{1,0}b_{1,3}+c_{4,19}^2b_{1,0}^2+c_{4,18}c_{4,19}b_{1,3}^2+c_{4,18}c_{4,19}b_{1,2}b_{1,3}+c_{4,18}c_{4,19}b_{1,2}^2+c_{4,18}c_{4,19}b_{1,1}^2+c_{4,18}c_{4,19}b_{1,0}b_{1,1}+c_{4,18}c_{4,19}b_{1,0}^2+c_{4,18}^2b_{1,2}^2+c_{4,18}^2b_{1,0}b_{1,3}+c_{4,18}^2b_{1,0}b_{1,2}+c_{4,18}^2b_{1,0}b_{1,1}+c_{4,18}^2b_{1,0}^2$

5.1.3 Groups of order 128

1. $SmallGroup(128,67)\cong Syl_2(U_3(7))$

   Minimal generating set: $a_{1,0},b_{1,1},a_{2,1},b_{2,2},a_{3,3},c_{4,4}$.

   $Sq^1a_{2,1}=a_{2,1}b_{1,1}$

   $Sq^1b_{2,2}=0$

   $Sq^1a_{3,3}=0$

$$Sq^2 a_{3,3} = b_{2,2}a_{3,3} + b_{2,2}^2 a_{1,0} + c_{4,4}a_{1,0}$$

$$Sq^1 c_{4,4} = 0$$

$$Sq^2 c_{4,4} = b_{2,2}^3 + c_{4,4}b_{1,1}^2 + b_{2,2}c_{4,4}$$

2. $SmallGroup(128, 147) \cong Syl_2(2PGU_2(31))$

   Minimal generating set: $a_{1,0}, b_{1,1}, b_{2,1}, c_{2,2}, c_{2,3}$.

$$Sq^1 b_{2,1} = b_{2,1}b_{1,1} + c_{2,2}b_{1,1}$$

$$Sq^1 c_{2,2} = 0$$

$$Sq^1 c_{2,3} = c_{2,3}b_{1,1} + c_{2,3}a_{1,0}$$

3. $SmallGroup(128, 928) \cong Syl_2(\mathcal{S}_8)$

   Minimal generating set:

$$b_{1,0}, b_{1,1}, b_{1,2}, b_{2,4}, b_{2,5}, b_{2,6}, b_{3,11}, b_{3,12}, c_{4,21}$$

$$Sq^1 b_{2,4} = b_{2,4}b_{1,1} + b_{2,4}b_{1,0}$$

$$Sq^1 b_{2,5} = b_{2,5}b_{1,2} + b_{2,5}b_{1,0}$$

$$Sq^1 b_{2,6} = b_{3,12} + b_{3,11} + b_{2,6}b_{1,1}$$

$$Sq^1 b_{3,11} = b_{1,1}b_{3,12} + b_{1,1}b_{3,11}$$

$$Sq^2 b_{3,11} = b_{1,1}^2 b_{3,12} + b_{2,6}b_{3,11} + b_{2,4}b_{3,11} + c_{4,21}b_{1,1}$$

$$Sq^1 b_{3,12} = 0$$

$$Sq^2 b_{3,12} = b_{1,2}^2 b_{3,12} + b_{2,6}b_{3,12} + b_{2,5}b_{3,12} + b_{2,5}b_{2,6}b_{1,2} + c_{4,21}b_{1,2}$$

$$Sq^1 c_{4,21} = c_{4,21}b_{1,2} + c_{4,21}b_{1,1}$$

$$Sq^2 c_{4,21} = c_{4,21}b_{1,1}b_{1,2} + c_{4,21}b_{1,0}^2 + b_{2,6}c_{4,21} + b_{2,5}c_{4,21} + b_{2,4}c_{4,21}$$

4. $SmallGroup(128, 931) \cong Syl_2(M_{22}) \cong Syl_2(M_{23})$

Minimal generating set:

$$b_{1,0}, b_{1,1}, b_{1,2}, b_{2,4}, b_{2,5}, b_{3,8}, b_{4,9}, b_{4,10}, b_{4,13}, b_{5,17}, b_{5,20}, b_{5,21},$$

$$b_{6,30}, b_{6,31}, b_{7,41}, b_{8,52}, b_{8,54}, c_{8,55}.$$

$Sq^1 b_{2,4} = b_{2,4}b_{1,1} + b_{2,4}b_{1,0}$

$Sq^1 b_{2,5} = b_{2,5}b_{1,2} + b_{2,5}b_{1,0}$

$Sq^1 b_{3,8} = 0$

$Sq^2 b_{3,8} = b_{5,21} + b_{5,17} + b_{1,1}^2 b_{3,8} + b_{4,10}b_{1,2} + b_{4,9}b_{1,2} + b_{2,4}b_{3,8}$

$Sq^1 b_{4,9} = b_{4,9}b_{1,2} + b_{4,9}b_{1,1} + b_{2,4}b_{3,8}$

$Sq^2 b_{4,9} = b_{1,2}b_{5,21} + b_{4,13}b_{1,1}^2 + b_{2,4}b_{4,13}$

$Sq^1 b_{4,10} = b_{4,10}b_{1,2} + b_{2,5}b_{3,8}$

$Sq^2 b_{4,10} = b_{1,2}b_{5,21} + b_{4,13}b_{1,2}^2 + b_{4,10}b_{1,2}^2 + b_{4,9}b_{1,2}^2 + b_{2,5}b_{4,13} + b_{2,5}b_{4,10} + b_{2,5}b_{4,9}$

$Sq^1 b_{4,13} = b_{4,9}b_{1,1} + b_{2,4}b_{3,8}$

$Sq^2 b_{4,13} = b_{4,13}b_{1,2}^2 + b_{4,9}b_{1,1}^2 + b_{2,5}b_{4,13} + b_{2,4}b_{4,9}$

$Sq^1 b_{5,17} = b_{1,0}b_{5,17} + b_{2,5}b_{4,9} + b_{2,4}b_{1,1}b_{3,8}$

$Sq^2 b_{5,17} = b_{6,30}b_{1,0} + b_{4,13}b_{1,2}^3 + b_{4,13}b_{1,1}^3 + b_{4,9}b_{1,1}^3 + b_{2,5}b_{5,17} + b_{2,5}b_{4,13}b_{1,2} + b_{2,5}b_{4,9}b_{1,2} +$
$b_{2,4}b_{5,21} + b_{2,4}b_{5,20} + b_{2,4}b_{5,17} + b_{2,4}b_{4,13}b_{1,2} + b_{2,4}b_{4,13}b_{1,1} + b_{2,4}^2 b_{3,8}$

$Sq^4 b_{5,17} = b_{6,30}b_{1,0}^3 + b_{4,10}b_{1,2}^5 + b_{4,10}^2 b_{1,2} + b_{4,9}b_{1,2}^5 + b_{4,9}b_{1,1}^5 + b_{4,9}^2 b_{1,2} + b_{4,9}^2 b_{1,1} +$
$b_{2,5}b_{6,30}b_{1,0} + b_{2,5}b_{4,13}b_{1,2}^3 + b_{2,5}b_{4,9}b_{1,2}^3 + b_{2,5}^2 b_{5,17} + b_{2,5}^2 b_{4,10}b_{1,2} + b_{2,4}b_{1,1}^4 b_{3,8} + b_{2,4}^2 b_{5,17} +$
$b_{2,4}b_{6,30}b_{1,0} + b_{2,4}b_{4,13}b_{1,1}^3 + b_{2,4}b_{4,9}b_{1,1}^3 + b_{2,4}^2 b_{1,1}^2 b_{3,8} + b_{2,4}^2 b_{4,9}b_{1,1} + c_{8,55}b_{1,0}$

$Sq^1 b_{5,20} = b_{1,1}b_{5,20} + b_{1,1}^3 b_{3,8} + b_{2,5}b_{4,9} + b_{2,4}b_{4,13} + b_{2,4}b_{4,9}$

$Sq^2 b_{5,20} = b_{4,13}b_{1,2}^3 + b_{2,5}b_{4,13}b_{1,2} + b_{2,5}b_{4,10}b_{1,2}$

$Sq^4 b_{5,20} = b_{1,1}^6 b_{3,8} + b_{6,30}b_{3,8} + b_{4,13}b_{5,20} + b_{4,13}b_{1,1}^2 b_{3,8} + b_{4,13}^2 b_{1,1} + b_{4,10}b_{5,20} + b_{4,10}b_{1,2}^5 +$
$b_{4,10}b_{4,13}b_{1,2} + b_{4,9}b_{5,20} + b_{4,9}b_{1,2}^5 + b_{4,9}b_{4,13}b_{1,1} + b_{4,9}b_{4,10}b_{1,2} + b_{4,9}^2 b_{1,2} + b_{2,5}b_{4,13}b_{1,2}^3 +$
$b_{2,5}b_{4,10}b_{3,8} + b_{2,5}b_{4,9}b_{1,2}^3 + b_{2,5}^2 b_{4,9}b_{1,2} + b_{2,4}b_{7,41} + b_{2,4}b_{4,13}b_{3,8} + b_{2,4}b_{4,13}b_{1,1}^3 + b_{2,4}^2 b_{5,17} +$

$b_{2,4}b_{4,9}b_{1,1}^3 + b_{2,4}^3b_{3,8} + c_{8,55}b_{1,1}$

$Sq^1b_{5,21} = b_{1,2}b_{5,21} + b_{1,1}b_{5,21} + b_{1,0}b_{5,17} + b_{4,13}b_{1,1}b_{1,2} + b_{4,9}b_{1,2}^2 + b_{4,9}b_{1,1}^2 + b_{2,5}b_{4,13} + b_{2,4}b_{1,1}b_{3,8} + b_{2,4}b_{4,13} + b_{2,4}b_{4,9}$

$Sq^2b_{5,21} = b_{1,1}^2b_{5,20} + b_{1,1}^4b_{3,8} + b_{6,30}b_{1,0} + b_{4,13}b_{1,1}^3 + b_{4,9}b_{1,1}^3 + b_{2,5}b_{5,17} + b_{2,5}b_{4,10}b_{1,2} + b_{2,5}b_{4,9}b_{1,2} + b_{2,4}b_{5,21} + b_{2,4}b_{5,17} + b_{2,4}b_{4,13}b_{1,2} + b_{2,4}b_{4,13}b_{1,1}$

$Sq^4b_{5,21} = b_{6,30}b_{3,8} + b_{6,30}b_{1,1}^3 + b_{6,30}b_{1,0}^3 + b_{4,13}b_{5,21} + b_{4,13}^2b_{1,2} + b_{4,13}^2b_{1,1} + b_{4,10}b_{5,21} + b_{4,10}b_{1,2}^5 + b_{4,10}b_{4,13}b_{1,2} + b_{4,10}^2b_{1,2} + b_{4,9}b_{5,20} + b_{4,9}b_{1,2}^5 + b_{4,9}b_{1,1}^2b_{3,8} + b_{4,9}b_{4,10}b_{1,2} + b_{4,9}^2b_{1,2} + b_{4,9}^2b_{1,1} + b_{2,5}b_{6,30}b_{1,0} + b_{2,5}b_{4,10}b_{3,8} + b_{2,5}b_{4,10}b_{1,2}^3 + b_{2,5}b_{4,9}b_{3,8} + b_{2,5}b_{4,9}b_{1,2}^3 + b_{2,5}^2b_{5,17} + b_{2,5}^2b_{4,13}b_{1,2} + b_{2,5}^2b_{4,10}b_{1,2} + b_{2,5}^2b_{4,9}b_{1,2} + b_{2,5}^3b_{3,8} + b_{2,4}b_{7,41} + b_{2,4}b_{6,30}b_{1,0} + b_{2,4}b_{4,9}b_{1,1}^3 + b_{2,4}^2b_{5,20} + b_{2,4}^2b_{4,13}b_{1,1} + b_{2,4}^3b_{3,8} + c_{8,55}b_{1,2} + c_{8,55}b_{1,1} + c_{8,55}b_{1,0}$

$Sq^1b_{6,30} = b_{4,10}b_{1,2}^3 + b_{4,9}b_{1,2}^3 + b_{4,9}b_{1,1}^3 + b_{2,5}b_{4,9}b_{1,2} + b_{2,5}^2b_{3,8} + b_{2,4}b_{5,21} + b_{2,4}b_{5,20} + b_{2,4}b_{4,13}b_{1,2} + b_{2,4}b_{4,13}b_{1,1} + b_{2,4}^2b_{3,8}$

$Sq^2b_{6,30} = b_{1,2}b_{7,41} + b_{8,52} + b_{6,30}b_{1,0}^2 + b_{4,13}b_{1,1}b_{3,8} + b_{4,10}b_{4,13} + b_{4,10}^2 + b_{4,9}b_{1,2}b_{3,8} + b_{4,9}b_{1,1}b_{3,8} + b_{4,9}^2 + b_{2,5}b_{4,10}b_{1,2}^2 + b_{2,4}b_{1,1}b_{5,20} + b_{2,4}b_{6,30} + b_{2,4}b_{4,13}b_{1,1}^2 + b_{2,4}^2b_{1,1}b_{3,8} + b_{2,4}^2b_{4,13}$

$Sq^4b_{6,30} = b_{6,30}b_{1,1}b_{3,8} + b_{4,13}b_{1,1}b_{5,21} + b_{4,13}b_{1,1}b_{5,20} + b_{4,13}^2b_{1,2}^2 + b_{4,13}^2b_{1,1}^2 + b_{4,10}b_{1,2}^6 + b_{4,10}^2b_{1,2}^2 + b_{4,9}b_{1,2}^6 + b_{4,9}b_{1,1}b_{5,20} + b_{4,9}b_{1,1}^6 + b_{4,9}^2b_{1,2}^2 + b_{2,5}b_{6,30}b_{1,0}^2 + b_{2,5}b_{4,10}^2 + b_{2,5}b_{4,9}b_{1,2}^4 + b_{2,5}^2b_{1,2}b_{5,21} + b_{2,5}^2b_{6,30} + b_{2,5}^2b_{4,9}b_{1,2}^2 + b_{2,5}^3b_{4,13} + b_{2,4}b_{1,1}^5b_{3,8} + b_{2,4}b_{6,30}b_{1,1}^2 + b_{2,4}b_{6,30}b_{1,0}^2 + b_{2,4}b_{4,13}^2 + b_{2,4}b_{4,9}b_{1,1}b_{3,8} + b_{2,4}^2b_{6,31} + b_{2,4}^2b_{4,13}b_{1,1}^2 + b_{2,4}^2b_{4,9}b_{1,1}^2 + b_{2,4}^3b_{1,1}b_{3,8} + b_{2,4}^3b_{4,13} + c_{8,55}b_{1,1}b_{1,2} + c_{8,55}b_{1,1}^2 + c_{8,55}b_{1,0}^2 + b_{2,4}c_{8,55}$

$Sq^1b_{6,31} = b_{4,10}b_{1,2}^3 + b_{4,9}b_{1,1}^3 + b_{2,5}b_{5,17} + b_{2,5}b_{4,10}b_{1,2} + b_{2,5}^2b_{3,8} + b_{2,4}b_{5,17} + b_{2,4}b_{1,1}^2b_{3,8} + b_{2,4}b_{4,13}b_{1,1}$

$Sq^2b_{6,31} = b_{1,1}b_{7,41} + b_{8,54} + b_{6,30}b_{1,1}^2 + b_{4,13}^2 + b_{4,10}b_{1,2}^4 + b_{4,10}b_{4,13} + b_{4,9}b_{1,1}^4 + b_{4,9}^2 + b_{2,5}b_{1,2}b_{5,21} + b_{2,5}b_{6,30} + b_{2,5}b_{4,13}b_{1,2}^2 + b_{2,5}b_{4,10}b_{1,2}^2 + b_{2,5}^2b_{4,10} + b_{2,5}^2b_{4,9} + b_{2,4}b_{1,1}b_{5,20} +$

$b_{2,4}b_{6,31} + b_{2,4}b_{4,9}b_{1,1}^2 + b_{2,4}^2b_{4,13}$

$Sq^4b_{6,31} = b_{6,30}b_{1,1}b_{3,8} + b_{4,13}b_{1,1}b_{5,21} + b_{4,13}b_{1,1}b_{5,20} + b_{4,10}b_{4,13}b_{1,2}^2 + b_{4,9}b_{1,2}b_{5,21} + b_{4,9}b_{1,1}b_{5,20} + b_{4,9}b_{1,1}^6 + b_{4,9}b_{4,13}b_{1,2}^2 + b_{4,9}b_{4,10}b_{1,2}^2 + b_{2,5}b_{8,52} + b_{2,5}b_{4,10}b_{4,13} + b_{2,5}b_{4,9}b_{1,2}^4 + b_{2,5}^2b_{6,30} + b_{2,5}^2b_{4,9}b_{1,2}^2 + b_{2,5}^3b_{4,13} + b_{2,5}^3b_{4,10} + b_{2,4}b_{1,1}^5b_{3,8} + b_{2,4}b_{6,30}b_{1,1}^2 + b_{2,4}b_{6,30}b_{1,0}^2 + b_{2,4}b_{4,13}^2 + b_{2,4}b_{4,9}b_{1,1}b_{3,8} + b_{2,4}b_{4,9}b_{1,1}^4 + b_{2,4}b_{4,9}^2 + b_{2,4}^2b_{1,1}^3b_{3,8} + b_{2,4}^2b_{6,31} + b_{2,4}^2b_{4,9}b_{1,1}^2 + b_{2,4}^3b_{4,9} + c_{8,55}b_{1,2}^2 + c_{8,55}b_{1,1}b_{1,2} + c_{8,55}b_{1,1}^2 + b_{2,5}c_{8,55} + b_{2,4}c_{8,55}$

$Sq^1b_{7,41} = b_{4,10}^2 + b_{4,9}b_{1,2}b_{3,8} + b_{4,9}b_{1,1}b_{3,8} + b_{4,9}b_{4,10} + b_{4,9}^2 + b_{2,5}b_{1,2}b_{5,21} + b_{2,5}b_{4,10}b_{1,2}^2 + b_{2,5}b_{4,9}b_{1,2}^2 + b_{2,5}^2b_{4,13} + b_{2,5}^2b_{4,9} + b_{2,4}b_{1,1}b_{5,20} + b_{2,4}b_{1,1}^3b_{3,8} + b_{2,4}b_{4,9}b_{1,1}^2 + b_{2,4}^2b_{1,1}b_{3,8} + b_{2,4}^2b_{4,13} + b_{2,4}^2b_{4,9}$

$Sq^2b_{7,41} = b_{6,30}b_{3,8} + b_{6,30}b_{1,1}^3 + b_{4,13}b_{5,21} + b_{4,13}b_{5,20} + b_{4,10}b_{5,20} + b_{4,10}^2b_{1,2} + b_{4,9}b_{5,21} + b_{4,9}b_{1,2}^5 + b_{4,9}b_{1,1}^2b_{3,8} + b_{4,9}b_{1,1}^5 + b_{4,9}^2b_{1,2} + b_{2,5}b_{4,13}b_{3,8} + b_{2,5}b_{4,13}b_{1,2}^3 + b_{2,5}b_{4,10}b_{1,2}^3 + b_{2,5}^2b_{5,21} + b_{2,5}^2b_{5,17} + b_{2,5}^2b_{4,13}b_{1,2} + b_{2,5}^2b_{4,10}b_{1,2} + b_{2,5}^2b_{4,9}b_{1,2} + b_{2,5}^3b_{3,8} + b_{2,4}b_{7,41} + b_{2,4}b_{1,1}^4b_{3,8} + b_{2,4}b_{6,30}b_{1,1} + b_{2,4}b_{4,9}b_{3,8} + b_{2,4}^2b_{5,17} + b_{2,4}^2b_{4,13}b_{1,1} + c_{8,55}b_{1,2} + c_{8,55}b_{1,1}$

$Sq^4b_{7,41} = b_{4,13}b_{7,41} + b_{4,13}b_{1,1}^2b_{5,20} + b_{4,13}b_{6,30}b_{1,1} + b_{4,13}^2b_{3,8} + b_{4,13}^2b_{1,2}^3 + b_{4,10}b_{7,41} + b_{4,10}b_{4,13}b_{3,8} + b_{4,10}^2b_{1,2}^3 + b_{4,9}b_{1,2}^7 + b_{4,9}b_{4,13}b_{1,1}^3 + b_{4,9}b_{4,10}b_{1,2}^3 + b_{4,9}^2b_{3,8} + b_{4,9}^2b_{1,1}^3 + b_{2,5}b_{4,13}b_{5,21} + b_{2,5}b_{4,13}^2b_{1,2} + b_{2,5}b_{4,10}b_{5,21} + b_{2,5}b_{4,10}b_{4,13}b_{1,2} + b_{2,5}b_{4,10}^2b_{1,2} + b_{2,5}b_{4,9}b_{5,21} + b_{2,5}^2b_{4,13}b_{3,8} + b_{2,5}^2b_{4,13}b_{1,2}^3 + b_{2,5}^2b_{4,10}b_{3,8} + b_{2,5}^2b_{4,10}b_{1,2}^3 + b_{2,5}^2b_{4,9}b_{1,2}^3 + b_{2,5}^3b_{5,21} + b_{2,5}^3b_{5,17} + b_{2,5}^3b_{4,10}b_{1,2} + b_{2,5}^4b_{3,8} + b_{2,4}b_{4,13}b_{5,21} + b_{2,4}b_{4,13}b_{5,20} + b_{2,4}b_{4,13}^2b_{1,2} + b_{2,4}b_{4,9}b_{5,20} + b_{2,4}b_{4,9}b_{1,1}^2b_{3,8} + b_{2,4}b_{4,9}b_{1,1}^5 + b_{2,4}^2b_{7,41} + b_{2,4}^2b_{6,30}b_{1,0} + b_{2,4}^2b_{4,9}b_{3,8} + b_{2,4}^3b_{5,17} + b_{2,4}^3b_{4,9}b_{1,1} + c_{8,55}b_{3,8} + c_{8,55}b_{1,2}^3 + b_{2,4}c_{8,55}b_{1,0}$

$Sq^1b_{8,52} = b_{4,10}b_{5,20} + b_{4,10}b_{1,2}^5 + b_{4,10}^2b_{1,2} + b_{4,9}b_{5,21} + b_{4,9}b_{1,2}^5 + b_{4,9}b_{4,13}b_{1,2} + b_{4,9}^2b_{1,2} + b_{4,9}^2b_{1,1} + b_{2,5}b_{6,30}b_{1,0} + b_{2,5}b_{4,13}b_{3,8} + b_{2,5}b_{4,13}b_{1,2}^3 + b_{2,5}b_{4,9}b_{1,2}^3 + b_{2,5}^2b_{4,9}b_{1,2} + b_{2,4}b_{7,41} + b_{2,4}b_{6,30}b_{1,1} + b_{2,4}b_{4,13}b_{3,8} + b_{2,4}b_{4,13}b_{1,1}^3 + b_{2,4}^2b_{5,20} + b_{2,4}^2b_{5,17} + b_{2,4}^2b_{4,13}b_{1,1} + b_{2,4}^2b_{4,9}b_{1,1} + b_{4,9}b_{4,10}b_{1,2}$

$Sq^2 b_{8,52} = b_{6,30}b_{1,0}^4 + b_{4,13}b_{1,1}b_{5,20} + b_{4,13}^2 b_{1,2}^2 + b_{4,13}^2 b_{1,1}b_{1,2} + b_{4,10}b_{4,13}b_{1,2}^2 + b_{4,10}^2 b_{1,2}^2 + b_{4,9}b_{1,2}b_{5,21} + b_{4,9}b_{1,2}^6 + b_{4,9}b_{1,1}^3 b_{3,8} + b_{4,9}b_{6,30} + b_{4,9}^2 b_{1,1}^2 + b_{2,5}b_{6,30}b_{1,0}^2 + b_{2,5}b_{4,13}^2 + b_{2,5}b_{4,10}b_{1,2}^4 + b_{2,5}^2 b_{1,2}b_{5,21} + b_{2,5}^2 b_{4,13}b_{1,2}^2 + b_{2,5}^2 b_{4,9}b_{1,2}^2 + b_{2,5}^3 b_{4,9} + b_{2,4}b_{1,1}^5 b_{3,8} + b_{2,4}b_{4,9}b_{1,1}^4 + b_{2,4}b_{4,9}^2 + b_{2,4}^2 b_{6,31} + b_{2,4}^2 b_{4,13}b_{1,1}^2 + b_{2,4}^2 b_{4,9}b_{1,1}^2 + b_{2,4}^3 b_{4,9} + c_{8,55}b_{1,2}^2 + c_{8,55}b_{1,1}b_{1,2} + b_{2,4}b_{6,30}b_{1,1}^2 + b_{2,4}b_{4,9}b_{1,1}b_{3,8}$

$Sq^4 b_{8,52} = b_{4,13}b_{8,52} + b_{4,10}^2 b_{1,2}^4 + b_{4,10}^2 b_{4,13} + b_{4,9}b_{1,2}^8 + b_{4,9}b_{8,54} + b_{4,9}b_{8,52} + b_{4,9}b_{6,30}b_{1,1}^2 + b_{4,9}b_{4,13}b_{1,2}b_{3,8} + b_{4,9}b_{4,10}b_{4,13} + b_{4,9}b_{4,10}^2 + b_{4,9}^2 b_{1,2}b_{3,8} + b_{4,9}^2 b_{1,2}^4 + b_{4,9}^2 b_{1,1}b_{3,8} + b_{4,9}^2 b_{1,1}^4 + b_{2,5}b_{4,13}b_{1,2}b_{5,21} + b_{2,5}b_{4,13}^2 b_{1,2}^2 + b_{2,5}b_{4,10}^2 b_{1,2}^2 + b_{2,5}b_{4,9}b_{1,2}b_{5,21} + b_{2,5}b_{4,9}b_{6,31} + b_{2,5}^2 b_{8,52} + b_{2,5}^2 b_{4,13}^2 + b_{2,5}^2 b_{4,10}^2 + b_{2,5}^2 b_{4,9}b_{1,2}^4 + b_{2,5}^2 b_{4,9}b_{4,13} + b_{2,5}^2 b_{4,9}b_{4,10} + b_{2,5}^2 b_{4,9}^2 + b_{2,5}^3 b_{1,2}b_{5,21} + b_{2,5}^3 b_{6,30} + b_{2,5}^3 b_{4,9}b_{1,2}^2 + b_{2,5}^4 b_{4,13} + b_{2,5}^4 b_{4,10} + b_{2,4}b_{1,1}^7 b_{3,8} + b_{2,4}b_{6,30}b_{1,1}b_{3,8} + b_{2,4}b_{6,30}b_{1,0}^4 + b_{2,4}b_{4,13}b_{1,1}b_{5,20} + b_{2,4}b_{4,13}b_{6,30} + b_{2,4}b_{4,9}b_{1,1}b_{5,20} + b_{2,4}b_{4,9}b_{6,30} + b_{2,4}b_{4,9}b_{4,13}b_{1,1}^2 + b_{2,4}^2 b_{8,54} + b_{2,4}^2 b_{6,30}b_{1,0}^2 + b_{2,4}^2 b_{4,9}^2 + b_{2,4}^3 b_{1,1}b_{5,20} + b_{2,4}^4 b_{4,13} + b_{2,4}^3 b_{4,13}b_{1,1}^2 + b_{2,4}^4 b_{1,1}b_{3,8} + c_{8,55}b_{1,2}^4 + c_{8,55}b_{1,0}^4 + b_{4,9}c_{8,55} + b_{2,5}c_{8,55}b_{1,2}^2 + b_{2,5}c_{8,55}b_{1,0}^2 + b_{2,4}c_{8,55}b_{1,1}^2 + b_{2,4}^2 c_{8,55} + b_{2,5}b_{4,9}^2 b_{1,2}^2 + b_{2,5}b_{4,9}b_{4,13}b_{1,2}^2$

$Sq^1 b_{8,54} = b_{6,30}b_{3,8} + b_{4,10}b_{5,21} + b_{4,10}b_{1,2}^5 + b_{4,10}b_{4,13}b_{1,2} + b_{4,9}b_{5,21} + b_{4,9}b_{5,20} + b_{4,9}b_{4,13}b_{1,1} + b_{4,9}b_{4,10}b_{1,2} + b_{4,9}^2 b_{1,2} + b_{2,5}b_{7,41} + b_{2,5}b_{4,13}b_{3,8} + b_{2,5}b_{4,10}b_{3,8} + b_{2,5}b_{4,10}b_{1,2}^3 + b_{2,5}b_{4,9}b_{3,8} + b_{2,5}^2 b_{5,21} + b_{2,5}^2 b_{5,17} + b_{2,5}^2 b_{4,10}b_{1,2} + b_{2,5}^2 b_{4,9}b_{1,2} + b_{2,5}^3 b_{3,8} + b_{2,4}b_{6,30}b_{1,1} + b_{2,4}b_{6,30}b_{1,0} + b_{2,4}b_{4,9}b_{3,8} + b_{2,4}b_{4,9}b_{1,1}^3 + b_{2,4}^2 b_{5,17} + b_{2,4}^2 b_{1,1}^2 b_{3,8} + b_{2,4}^2 b_{4,13}b_{1,1} + b_{2,4}^2 b_{4,9}b_{1,1} + b_{2,4}^3 b_{3,8}$

$Sq^2 b_{8,54} = b_{6,30}b_{1,1}b_{3,8} + b_{4,13}b_{1,2}b_{5,21} + b_{4,13}b_{1,1}b_{5,21} + b_{4,13}b_{1,1}b_{5,20} + b_{4,13}^2 b_{1,2}^2 + b_{4,10}b_{1,2}^6 + b_{4,9}b_{1,1}b_{5,20} + b_{4,9}b_{1,1}^3 b_{3,8} + b_{4,9}b_{6,30} + b_{4,9}b_{4,10}b_{1,2}^2 + b_{2,5}b_{4,13}^2 + b_{2,5}b_{4,9}b_{4,13} + b_{2,5}b_{4,9}b_{4,10} + b_{2,5}^2 b_{1,2}b_{5,21} + b_{2,5}^2 b_{4,13}b_{1,2}^2 + b_{2,5}^3 b_{4,13} + b_{2,4}b_{8,54} + b_{2,4}b_{8,52} + b_{2,4}b_{6,30}b_{1,1}^2 + b_{2,4}b_{6,30}b_{1,0}^2 + b_{2,4}b_{4,13}b_{1,1}b_{3,8} + b_{2,4}b_{4,13}^2 + b_{2,4}b_{4,9}b_{1,1}b_{3,8} + b_{2,4}b_{4,9}b_{1,1}^4 + b_{2,4}b_{4,9}^2 + b_{2,4}^2 b_{6,31} + b_{2,4}^3 b_{1,1}b_{3,8} + b_{2,4}^3 b_{4,13} + c_{8,55}b_{1,1}b_{1,2} + c_{8,55}b_{1,1}^2$

$Sq^4 b_{8,54} = b_{6,30}b_{1,1}^6 + b_{4,13}b_{8,54} + b_{4,13}^2 b_{1,1}b_{3,8} + b_{4,13}^3 + b_{4,10}b_{4,13}^2 + b_{4,10}^2 b_{4,13} + b_{4,9}b_{1,1}^5 b_{3,8} + b_{4,9}b_{1,2}b_{7,41} + b_{4,9}b_{1,1}^8 + b_{4,9}b_{8,54} + b_{4,9}b_{4,13}b_{1,2}b_{3,8} + b_{4,9}b_{4,13}b_{1,1}b_{3,8} + b_{4,9}b_{4,10}b_{1,2}^4 + b_{4,9}b_{4,10}b_{4,13} + b_{4,9}^2 b_{4,13} + b_{4,9}^3 + b_{2,5}b_{4,13}b_{6,31} + b_{2,5}b_{4,10}^2 b_{1,2}^2 + b_{2,5}b_{4,9}b_{1,2}^6 + b_{2,5}b_{4,9}b_{4,13}b_{1,2}^2 + b_{2,5}^2 b_{8,52} + b_{2,5}^2 b_{6,30}b_{1,0}^2 + b_{2,5}^2 b_{4,13}^2 + b_{2,5}^2 b_{4,10}b_{4,13} + b_{2,5}^2 b_{4,10}^2 + b_{2,5}^2 b_{4,9}b_{1,2}^4 + b_{2,5}^2 b_{4,9}b_{4,13} + b_{2,5}^3 b_{1,2}b_{5,21} + b_{2,5}^3 b_{6,30} + b_{2,5}^3 b_{4,13}b_{1,2}^2 + b_{2,5}^3 b_{4,9}b_{1,2}^2 + b_{2,5}^4 b_{4,13} + b_{2,5}^4 b_{4,10} + b_{2,4}b_{6,30}b_{1,1}b_{3,8} + b_{2,4}b_{1,1}^7 b_{3,8} + b_{2,4}b_{6,30}b_{1,1}^4 + b_{2,4}b_{4,13}b_{1,1}b_{5,20} + b_{2,4}b_{4,9}b_{1,1}^3 b_{3,8} + b_{2,4}b_{4,9}b_{6,30} + b_{2,4}^2 b_{8,54} + b_{2,4}b_{4,9}b_{4,13}b_{1,1}^2 + b_{2,4}^2 b_{6,30}b_{1,1}^2 + b_{2,4}^2 b_{6,30}b_{1,0}^2 + b_{2,4}^2 b_{4,13}b_{1,1}b_{3,8} + b_{2,4}^2 b_{4,13}^2 + b_{2,4}^2 b_{4,9}b_{1,1}^4 + b_{2,4}^2 b_{4,9}^2 + b_{2,4}^3 b_{6,31} + b_{2,4}^3 b_{6,30} + b_{2,4}^3 b_{4,13}b_{1,1}^2 + b_{2,4}^4 b_{4,9} + c_{8,55}b_{1,2}^4 + c_{8,55}b_{1,1}b_{3,8} + b_{4,10}c_{8,55} + b_{4,9}c_{8,55} + b_{2,5}c_{8,55}b_{1,2}^2 + b_{2,4}c_{8,55}b_{1,0}^2 + b_{2,4}^2 c_{8,55}$

$Sq^1 c_{8,55} = b_{6,30}b_{3,8} + b_{4,10}b_{5,21} + b_{4,10}b_{1,2}^5 + b_{4,10}b_{4,13}b_{1,2} + b_{4,9}b_{1,1}^2 b_{3,8} + b_{4,9}b_{4,13}b_{1,2} + b_{4,9}b_{4,13}b_{1,1} + b_{4,9}^2 b_{1,2} + b_{4,9}^2 b_{1,1} + b_{2,5}b_{7,41} + b_{2,5}b_{4,10}b_{1,2}^3 + b_{2,5}^2 b_{4,9}b_{1,2} + b_{2,5}^3 b_{3,8} + b_{2,4}b_{6,30}b_{1,0} + b_{2,4}b_{4,13}b_{1,1}^3 + b_{2,4}b_{4,9}b_{1,1}^3 + b_{2,4}^2 b_{5,17} + b_{2,4}^3 b_{3,8}$

$Sq^2 c_{8,55} = b_{6,30}b_{1,1}b_{3,8} + b_{6,30}b_{1,0}^4 + b_{4,13}b_{1,2}b_{5,21} + b_{4,13}^2 b_{1,1}b_{1,2} + b_{4,10}b_{4,13}b_{1,2}^2 + b_{4,10}^2 b_{1,2}^2 + b_{4,9}b_{1,1}^3 b_{3,8} + b_{4,9}b_{6,31} + b_{4,9}b_{6,30} + b_{4,9}b_{4,13}b_{1,2}^2 + b_{4,9}^2 b_{1,2}^2 + b_{2,5}b_{8,54} + b_{2,5}b_{8,52} + b_{2,5}b_{4,13}^2 + b_{2,5}b_{4,9}b_{1,2}^4 + b_{2,5}b_{4,9}b_{4,13} + b_{2,5}b_{4,9}^2 + b_{2,5}^2 b_{6,31} + b_{2,5}^2 b_{6,30} + b_{2,5}^2 b_{4,13}b_{1,2}^2 + b_{2,5}^3 b_{4,13} + b_{2,5}^3 b_{4,10} + b_{2,5}^3 b_{4,9} + b_{2,4}b_{8,54} + b_{2,4}b_{8,52} + b_{2,4}b_{6,30}b_{1,1}^2 + b_{2,4}b_{6,30}b_{1,0}^2 + b_{2,4}b_{4,13}b_{1,1}b_{3,8} + b_{2,4}b_{4,13}^2 + b_{2,4}b_{4,9}b_{1,1}^4 + b_{2,4}^2 b_{1,1}b_{5,20} + b_{2,4}^2 b_{1,1}^3 b_{3,8} + b_{2,4}^2 b_{4,9}b_{1,1}^2 + b_{2,4}^3 b_{4,13} + c_{8,55}b_{1,2}^2 + c_{8,55}b_{1,1}^2$

$Sq^4 c_{8,55} = b_{6,30}b_{1,1}^6 + b_{6,30}^2 + b_{4,13}^2 b_{1,1}b_{3,8} + b_{4,13}^3 + b_{4,10}b_{4,13}^2 + b_{4,10}^2 b_{1,2}^4 + b_{4,9}b_{1,1}^5 b_{3,8} + b_{4,9}b_{1,1}^8 + b_{4,9}b_{4,13}b_{1,1}b_{3,8} + b_{4,9}^2 b_{1,2}b_{3,8} + b_{4,9}^2 b_{1,2}^4 + b_{4,9}^2 b_{1,1}^4 + b_{4,9}^2 b_{4,13} + b_{4,9}^2 b_{4,10} + b_{4,9}^3 + b_{2,5}b_{4,10}b_{1,2}^6 + b_{2,5}b_{4,10}b_{4,13}b_{1,2}^2 + b_{2,5}b_{4,10}^2 b_{1,2}^2 + b_{2,5}b_{4,9}b_{1,2}b_{5,21} + b_{2,5}b_{4,9}b_{6,31} + b_{2,5}b_{4,9}b_{4,10}b_{1,2}^2 + b_{2,5}b_{4,9}^2 b_{1,2}^2 + b_{2,5}^2 b_{8,52} + b_{2,5}^2 b_{6,30}b_{1,0}^2 + b_{2,5}^2 b_{4,13}^2 + b_{2,5}^2 b_{4,10}^2 + b_{2,5}^2 b_{4,9}^2 + b_{2,5}^3 b_{1,2}b_{5,21} + b_{2,5}^3 b_{6,31} + b_{2,5}^3 b_{4,9}b_{1,2}^2 + b_{2,5}^4 b_{4,13} + b_{2,5}^4 b_{4,9} + b_{2,4}b_{1,1}^7 b_{3,8} + b_{2,4}b_{6,30}b_{1,1}^4 + b_{2,4}b_{6,30}b_{1,0}^4 + b_{2,4}b_{4,13}b_{1,1}b_{5,20} + b_{2,4}b_{4,13}b_{6,30} + b_{2,4}b_{4,13}^2 b_{1,1}^2 + b_{2,4}b_{4,9}b_{1,1}^3 b_{3,8} + b_{2,4}^2 b_{8,54} +$

$b_{2,4}b_{4,9}b_{6,30}+b_{2,4}b_{4,9}b_{4,13}b_{1,1}^2+b_{2,4}b_{4,9}^2b_{1,1}^2+b_{2,4}^2b_{6,30}b_{1,1}^2+b_{2,4}^2b_{4,9}b_{1,1}b_{3,8}+b_{2,4}^2b_{4,9}b_{1,1}^4+$ $b_{2,4}^2b_{4,9}^2+b_{2,4}^3b_{6,31}+b_{2,4}^3b_{4,13}b_{1,1}^2+b_{2,4}^3b_{4,9}b_{1,1}^2+c_{8,55}b_{1,2}^4+c_{8,55}b_{1,1}b_{3,8}+c_{8,55}b_{1,0}^4+$ $b_{4,13}c_{8,55}+b_{4,10}c_{8,55}+b_{2,5}c_{8,55}b_{1,2}^2+b_{2,4}c_{8,55}b_{1,1}^2+b_{2,4}c_{8,55}b_{1,0}^2$

5. $SmallGroup(128,932) \cong Syl_2(G_2(3):2)$

   Minimal generating set:

$$b_{1,0}, b_{1,1}, b_{1,2}, b_{2,4}, b_{2,5}, b_{3,8}, b_{3,9}, b_{4,11}, b_{4,14}, b_{5,17}, b_{5,20}, b_{5,21}, b_{6,29}, b_{7,38}, b_{8,46}, c_{8,50}$$

$Sq^1b_{2,4} = b_{2,4}b_{1,1}+b_{2,4}b_{1,0}$

$Sq^1b_{2,5} = b_{2,5}b_{1,2}+b_{2,5}b_{1,0}$

$Sq^1b_{3,8} = b_{1,1}b_{3,9}$

$Sq^2b_{3,8} = b_{5,21}+b_{5,20}+b_{4,14}b_{1,2}+b_{4,11}b_{1,2}+b_{2,4}b_{1,1}^3$

$Sq^1b_{3,9} = b_{1,1}b_{3,9}$

$Sq^2b_{3,9} = b_{5,21}+b_{1,1}^2b_{3,8}+b_{1,1}^5+b_{2,5}b_{3,9}+b_{2,4}b_{3,9}+b_{2,4}b_{3,8}$

$Sq^1b_{4,11} = b_{4,11}b_{1,2}+b_{2,5}b_{3,9}$

$Sq^2b_{4,11} = b_{1,2}^3b_{3,9}+b_{4,14}b_{1,2}^2+b_{4,14}b_{1,1}b_{1,2}+b_{2,5}b_{1,2}b_{3,9}+b_{2,5}b_{4,14}+b_{2,4}b_{1,1}b_{3,9}+$ $b_{2,4}b_{1,1}b_{3,8}+b_{2,4}^2b_{1,1}^2$

$Sq^1b_{4,14} = b_{1,2}^2b_{3,9}+b_{4,11}b_{1,2}+b_{2,5}b_{3,9}+b_{2,4}b_{3,9}+b_{2,4}b_{3,8}+b_{2,4}b_{1,1}^3$

$Sq^2b_{4,14} = b_{1,2}b_{5,20}+b_{1,1}b_{5,21}+b_{1,1}b_{5,20}+b_{1,1}^3b_{3,8}+b_{4,14}b_{1,2}^2+b_{2,5}b_{1,2}b_{3,9}+b_{2,5}b_{4,11}+$ $b_{2,4}b_{1,1}b_{3,8}+b_{2,4}b_{1,1}^4+b_{2,4}b_{4,14}$

$Sq^1b_{5,17} = b_{1,2}^3b_{3,9}+b_{1,0}b_{5,17}+b_{4,14}b_{1,2}^2+b_{4,14}b_{1,1}^2+b_{4,11}b_{1,2}^2+b_{2,4}b_{1,1}b_{3,8}+b_{2,4}^2b_{1,1}^2$

$Sq^2b_{5,17} = b_{1,2}^4b_{3,9}+b_{6,29}b_{1,0}+b_{4,14}b_{1,2}^3+b_{4,14}b_{1,1}^3+b_{4,11}b_{1,2}^3+b_{2,5}b_{1,2}^2b_{3,9}+b_{2,5}b_{4,14}b_{1,2}+$ $b_{2,5}b_{4,11}b_{1,2}+b_{2,4}b_{5,17}+b_{2,4}b_{1,1}^2b_{3,8}+b_{2,4}b_{1,1}^5+b_{2,4}b_{4,14}b_{1,1}+b_{2,4}^3b_{1,1}+b_{2,4}^3b_{1,0}$

$Sq^4b_{5,17} = b_{1,2}^4b_{5,20}+b_{8,46}b_{1,1}+b_{6,29}b_{1,1}^3+b_{4,14}b_{5,17}+b_{4,14}b_{1,2}^2b_{3,9}+b_{4,14}b_{1,1}^2b_{3,8}+$ $b_{4,14}b_{1,1}^5+b_{4,11}b_{4,14}b_{1,2}+b_{2,5}b_{1,2}^4b_{3,9}+b_{2,5}b_{6,29}b_{1,0}+b_{2,5}b_{4,11}b_{1,2}^3+b_{2,5}^2b_{5,17}+b_{2,5}^2b_{1,2}^2b_{3,9}+$

$b_{2,4}b_{1,1}^7+b_{2,4}b_{6,29}b_{1,1}+b_{2,4}b_{4,14}b_{3,9}+b_{2,4}b_{4,14}b_{3,8}+b_{2,4}b_{4,14}b_{1,1}^3+b_{2,4}^2b_{4,14}b_{1,1}+b_{2,4}^4b_{1,1}+b_{2,4}^4b_{1,0}+c_{8,50}b_{1,0}$

$Sq^1b_{5,20}=b_{1,2}b_{5,20}+b_{2,5}b_{1,2}b_{3,9}+b_{2,5}b_{4,14}+b_{2,5}b_{4,11}$

$Sq^2b_{5,20}=b_{1,2}^2b_{5,20}+b_{2,5}b_{5,20}+b_{2,4}b_{5,20}+b_{2,4}b_{1,1}^5$

$Sq^4b_{5,20}=b_{1,2}^4b_{5,20}+b_{8,46}b_{1,1}+b_{4,14}b_{5,20}+b_{4,14}b_{1,2}^2b_{3,9}+b_{4,14}b_{1,1}^2b_{3,8}+b_{4,14}b_{1,1}^5+b_{4,14}^2b_{1,2}+b_{4,11}b_{5,20}+b_{4,11}b_{1,2}^2b_{3,9}+b_{4,11}^2b_{1,2}+b_{2,5}b_{1,2}^2b_{5,20}+b_{2,5}b_{4,14}b_{1,2}^3+b_{2,4}b_{7,38}+b_{2,4}b_{1,1}^2b_{5,20}+b_{2,4}b_{6,29}b_{1,1}+b_{2,4}b_{4,14}b_{3,8}+b_{2,4}^2b_{5,21}+b_{2,4}^2b_{5,17}+b_{2,4}^3b_{3,8}+b_{2,4}^4b_{1,1}+c_{8,50}b_{1,2}$

$Sq^1b_{5,21}=b_{1,2}b_{5,20}+b_{1,2}^3b_{3,9}+b_{1,1}b_{5,21}+b_{1,1}b_{5,20}+b_{1,1}^3b_{3,8}+b_{4,14}b_{1,2}^2+b_{4,14}b_{1,1}b_{1,2}+b_{4,11}b_{1,2}^2+b_{2,5}b_{1,2}b_{3,9}+b_{2,5}b_{4,14}+b_{2,5}b_{4,11}+b_{2,4}b_{1,1}b_{3,9}+b_{2,4}b_{1,1}^4+b_{2,4}b_{4,14}$

$Sq^2b_{5,21}=b_{1,1}^2b_{5,20}+b_{1,1}^7+b_{4,14}b_{1,1}^3+b_{2,5}b_{5,20}+b_{2,5}b_{4,14}b_{1,2}+b_{2,5}b_{4,11}b_{1,2}+b_{2,4}b_{4,14}b_{1,1}$

$Sq^4b_{5,21}=b_{1,2}^4b_{5,20}+b_{6,29}b_{1,1}^3+b_{4,14}b_{5,21}+b_{4,14}b_{5,17}+b_{4,14}b_{1,2}^5+b_{4,14}b_{1,1}^2b_{3,8}+b_{4,14}b_{1,1}^5+b_{4,11}b_{5,20}+b_{4,11}b_{1,2}^2b_{3,9}+b_{4,11}b_{1,2}^5+b_{4,11}^2b_{1,2}+b_{2,5}b_{1,2}^2b_{5,20}+b_{2,5}b_{4,14}b_{1,2}^3+b_{2,5}b_{4,11}b_{1,2}^3+b_{2,5}^2b_{4,14}b_{1,2}+b_{2,5}^2b_{4,11}b_{1,2}+b_{2,4}b_{1,1}^7+b_{2,4}b_{4,14}b_{3,8}+b_{2,4}b_{4,14}b_{1,1}^3+b_{2,4}^2b_{5,21}+b_{2,4}^4b_{1,1}+c_{8,50}b_{1,2}+c_{8,50}b_{1,1}$

$Sq^1b_{6,29}=b_{1,2}^4b_{3,9}+b_{6,29}b_{1,1}+b_{4,11}b_{1,2}^3+b_{2,5}b_{5,17}+b_{2,5}b_{4,14}b_{1,2}+b_{2,5}b_{4,11}b_{1,2}+b_{2,4}b_{4,14}b_{1,2}+b_{2,4}^3b_{1,0}$

$Sq^2b_{6,29}=b_{3,8}b_{5,21}+b_{1,2}b_{7,38}+b_{1,1}b_{7,38}+b_{1,1}^3b_{5,20}+b_{8,46}+b_{6,29}b_{1,0}^2+b_{4,14}b_{1,2}b_{3,9}+b_{4,14}b_{1,1}b_{3,9}+b_{4,14}b_{1,1}b_{3,8}+b_{4,14}b_{1,1}^4+b_{4,11}^2+b_{2,5}b_{1,2}^3b_{3,9}+b_{2,5}b_{1,0}b_{5,17}+b_{2,5}b_{4,14}b_{1,2}^2+b_{2,5}b_{4,11}b_{1,2}^2+b_{2,5}^2b_{4,14}+b_{2,4}b_{1,1}b_{5,21}+b_{2,4}b_{1,0}b_{5,17}+b_{2,4}b_{4,14}b_{1,1}^2+b_{2,4}^2b_{4,14}+b_{2,4}^3b_{1,1}^2$

$Sq^4b_{6,29}=b_{8,46}b_{1,2}^2+b_{8,46}b_{1,1}^2+b_{6,29}b_{1,1}^4+b_{6,29}b_{1,0}^4+b_{4,14}b_{1,2}b_{5,20}+b_{4,14}b_{1,2}^3b_{3,9}+b_{4,14}b_{1,2}^6+b_{4,14}b_{1,1}b_{5,20}+b_{4,14}b_{1,1}^6+b_{4,11}b_{1,2}b_{5,20}+b_{4,11}b_{1,2}^3b_{3,9}+b_{4,11}b_{1,2}^6+b_{4,11}b_{4,14}b_{1,2}^2+b_{4,11}^2b_{1,2}^2+b_{2,5}b_{8,46}+b_{2,5}b_{6,29}b_{1,0}^2+b_{2,5}b_{4,14}^2+b_{2,5}b_{4,11}b_{1,2}b_{3,9}+b_{2,5}b_{4,11}b_{4,14}+b_{2,5}^2b_{1,2}^3b_{3,9}+b_{2,5}^2b_{6,29}+b_{2,5}^2b_{4,14}b_{1,2}^2+b_{2,5}^3b_{1,2}b_{3,9}+b_{2,5}^3b_{4,14}+b_{2,4}b_{1,1}^8+b_{2,4}b_{4,14}b_{1,1}^4+b_{2,4}b_{4,14}^2+b_{2,4}^2b_{1,1}b_{5,21}+b_{2,4}^3b_{1,1}b_{3,8}+b_{2,4}^3b_{1,0}^4+b_{2,4}^3b_{4,14}+b_{2,4}^5+c_{8,50}b_{1,0}^2+b_{2,5}c_{8,50}$

$Sq^1 b_{7,38} = b_{1,2}^3 b_{5,20} + b_{4,14} b_{1,1}^4 + b_{4,11} b_{1,2} b_{3,9} + b_{4,11}^2 + b_{2,5} b_{1,2} b_{5,20} + b_{2,5} b_{1,2}^3 b_{3,9} + b_{2,5} b_{4,11} b_{1,2}^2 + b_{2,4} b_{1,1} b_{5,20} + b_{2,4} b_{1,1}^6 + b_{2,4} b_{1,0} b_{5,17} + b_{2,4} b_{6,29} + b_{2,4} b_{4,14} b_{1,1}^2 + b_{2,4}^2 b_{1,1} b_{3,8} + b_{2,4}^3 b_{1,1}^2 + b_{2,4}^4$

$Sq^2 b_{7,38} = b_{8,46} b_{1,2} + b_{8,46} b_{1,1} + b_{6,29} b_{1,1}^3 + b_{4,14} b_{5,21} + b_{4,14} b_{5,20} + b_{4,14} b_{1,1}^2 b_{3,8} + b_{4,14} b_{1,1}^5 + b_{4,14}^2 b_{1,2} + b_{4,14}^2 b_{1,1} + b_{4,11} b_{1,2}^2 b_{3,9} + b_{4,11}^2 b_{1,2} + b_{2,5}^2 b_{1,2}^2 b_{3,9} + b_{2,5}^2 b_{4,14} b_{1,2} + b_{2,4} b_{7,38} + b_{2,4} b_{1,1}^2 b_{5,20} + b_{2,4} b_{1,1}^7 + b_{2,4} b_{6,29} b_{1,1} + b_{2,4} b_{4,14} b_{1,1}^3 + b_{2,4}^2 b_{5,21} + b_{2,4}^2 b_{5,17} + b_{2,4}^2 b_{4,14} b_{1,1} + b_{2,4}^3 b_{3,8} + b_{2,4}^4 b_{1,1} + c_{8,50} b_{1,2}$

$Sq^4 b_{7,38} = b_{8,46} b_{1,1}^3 + b_{6,29} b_{5,20} + b_{6,29} b_{1,1}^2 b_{3,8} + b_{6,29} b_{1,1}^5 + b_{4,14} b_{7,38} + b_{4,14} b_{1,2}^2 b_{5,20} + b_{4,14} b_{1,2}^4 b_{3,9} + b_{4,14} b_{6,29} b_{1,1} + b_{4,14}^2 b_{3,9} + b_{4,14}^2 b_{3,8} + b_{4,14}^2 b_{1,1}^3 + b_{4,11} b_{1,2}^4 b_{3,9} + b_{4,11} b_{1,2}^7 + b_{4,11}^2 b_{3,9} + b_{4,11}^2 b_{1,2}^3 + b_{2,5} b_{1,2}^4 b_{5,20} + b_{2,5} b_{8,46} b_{1,2} + b_{2,5} b_{4,14} b_{5,20} + b_{2,5} b_{4,14} b_{1,2}^2 b_{3,9} + b_{2,5} b_{4,14} b_{1,2}^5 + b_{2,5} b_{4,14}^2 b_{1,2} + b_{2,5} b_{4,11} b_{1,2}^2 b_{3,9} + b_{2,5} b_{4,11} b_{4,14} b_{1,2} + b_{2,5}^2 b_{7,38} + b_{2,5}^2 b_{4,14} b_{3,9} + b_{2,5}^2 b_{4,11} b_{3,9} + b_{2,5}^2 b_{4,11} b_{1,2}^3 + b_{2,5}^3 b_{4,11} b_{1,2} + b_{2,4} b_{1,1}^2 b_{7,38} + b_{2,4} b_{8,46} b_{1,1} + b_{2,4} b_{6,29} b_{1,1}^3 + b_{2,4} b_{6,29} b_{1,0}^3 + b_{2,4} b_{4,14} b_{5,21} + b_{2,4} b_{4,14} b_{1,1}^2 b_{3,8} + b_{2,4}^2 b_{6,29} b_{1,0} + b_{2,4}^3 b_{5,21} + b_{2,4}^3 b_{5,17} + b_{2,4}^4 b_{3,8} + b_{2,4}^4 b_{1,0}^3 + b_{2,4}^5 b_{1,1} + c_{8,50} b_{3,9} + c_{8,50} b_{3,8} + b_{2,4} c_{8,50} b_{1,2} + b_{2,4} c_{8,50} b_{1,1} + b_{2,4} c_{8,50} b_{1,0}$

$Sq^1 b_{8,46} = b_{1,2}^4 b_{5,20} + b_{8,46} b_{1,2} + b_{4,14} b_{5,17} + b_{4,14} b_{1,2}^2 b_{3,9} + b_{4,14} b_{1,1}^5 + b_{4,14}^2 b_{1,2} + b_{4,14}^2 b_{1,1} + b_{4,11} b_{4,14} b_{1,2} + b_{4,11}^2 b_{1,2} + b_{2,5} b_{7,38} + b_{2,5} b_{1,2}^2 b_{5,20} + b_{2,5} b_{1,2}^4 b_{3,9} + b_{2,5} b_{4,11} b_{3,9} + b_{2,5}^2 b_{4,14} b_{1,2} + b_{2,5}^2 b_{4,11} b_{1,2} + b_{2,5}^3 b_{3,9} + b_{2,4} b_{7,38} + b_{2,4} b_{1,1}^2 b_{5,20} + b_{2,4} b_{1,1}^7 + b_{2,4} b_{4,14} b_{3,9} + b_{2,4} b_{4,14} b_{3,8} + b_{2,4}^2 b_{5,21} + b_{2,4}^2 b_{5,17} + b_{2,4}^3 b_{3,8} + b_{2,4}^4 b_{1,1}$

$Sq^2 b_{8,46} = b_{6,29} b_{1,0}^4 + b_{4,14} b_{1,2} b_{5,20} + b_{4,14} b_{1,2}^3 b_{3,9} + b_{4,14} b_{1,1}^6 + b_{4,14}^2 b_{1,2}^2 + b_{4,14}^2 b_{1,1} b_{1,2} + b_{4,14}^2 b_{1,1}^2 + b_{4,11} b_{1,2} b_{5,20} + b_{4,11} b_{1,2}^3 b_{3,9} + b_{4,11} b_{1,2}^6 + b_{4,11}^2 b_{1,2}^2 + b_{2,5} b_{1,2}^3 b_{5,20} + b_{2,5} b_{4,14} b_{1,2}^4 + b_{2,5} b_{4,11} b_{1,2}^4 + b_{2,5} b_{4,11}^2 + b_{2,5}^2 b_{4,11} b_{1,2}^2 + b_{2,5}^3 b_{1,2} b_{3,9} + b_{2,5}^3 b_{4,11} + b_{2,4} b_{6,29} b_{1,0}^2 + b_{2,4} b_{4,14}^2 + b_{2,4} b_{4,14} b_{1,1} b_{3,9} + b_{2,4} b_{4,14} b_{1,1} b_{3,8} + b_{2,4} b_{4,14} b_{1,1}^4 + b_{2,4}^2 b_{1,1} b_{5,21} + b_{2,4}^3 b_{1,0}^4 + b_{2,4}^4 b_{1,1}^2 + b_{2,4}^4 b_{1,0}^2 + c_{8,50} b_{1,2}^2 + c_{8,50} b_{1,1}^2$

$Sq^4 b_{8,46} = b_{8,46} b_{1,1}^4 + b_{6,29} b_{1,0}^6 + b_{4,14} b_{1,2}^3 b_{5,20} + b_{4,14} b_{1,1}^3 b_{5,20} + b_{4,14} b_{8,46} + b_{4,14} b_{6,29} b_{1,1}^2 +$

$b_{4,14}^2 b_{1,2} b_{3,9} + b_{4,14}^2 b_{1,1} b_{3,9} + b_{4,14}^2 b_{1,1}^4 + b_{4,11} b_{1,2}^3 b_{5,20} + b_{4,11} b_{1,2}^8 + b_{4,11} b_{8,46} + b_{4,11} b_{4,14} b_{1,2}^4 + b_{4,11} b_{4,14} b_{1,2} b_{3,9} + b_{4,11} b_{4,14}^2 + b_{4,11}^2 b_{1,2} b_{3,9} + b_{4,11}^2 b_{1,2}^4 + b_{4,11}^3 + b_{2,5} b_{8,46} b_{1,2}^2 + b_{2,5} b_{6,29} b_{1,0}^4 + b_{2,5} b_{4,14} b_{1,2}^3 b_{3,9} + b_{2,5} b_{4,14} b_{1,2}^6 + b_{2,5} b_{4,14}^2 b_{1,2}^2 + b_{2,5} b_{4,11} b_{1,2} b_{5,20} + b_{2,5} b_{4,11} b_{1,2}^3 b_{3,9} + b_{2,5} b_{4,11} b_{4,14} b_{1,2}^2 + b_{2,5} b_{4,11}^2 b_{1,2}^2 + b_{2,5}^2 b_{8,46} + b_{2,5}^2 b_{4,14} b_{1,2}^4 + b_{2,5}^2 b_{4,14}^2 + b_{2,5}^2 b_{4,11} b_{1,2} b_{3,9} + b_{2,5}^2 b_{4,11} b_{1,2}^4 + b_{2,5}^3 b_{1,2}^3 b_{3,9} + b_{2,5}^3 b_{4,14} b_{1,2}^2 + b_{2,5}^3 b_{4,11} b_{1,2}^2 + b_{2,4} b_{1,1}^3 b_{7,38} + b_{2,4} b_{6,29} b_{1,0}^4 + b_{2,4} b_{4,14}^2 b_{1,1}^2 + b_{2,4}^2 b_{8,46} + b_{2,4}^2 b_{6,29} b_{1,0}^2 + b_{2,4}^2 b_{4,14} b_{1,1} b_{3,8} + b_{2,4}^3 b_{1,0}^6 + b_{2,4}^4 b_{1,1} b_{3,8} + b_{2,4}^4 b_{1,0}^4 + b_{2,4}^4 b_{4,14} + b_{2,4}^5 b_{1,0}^2 + b_{2,4}^6 + c_{8,50} b_{1,2} b_{3,9} + c_{8,50} b_{1,2}^4 + c_{8,50} b_{1,1} b_{3,9} + c_{8,50} b_{1,0}^4 + b_{4,11} c_{8,50}$

$Sq^1 c_{8,50} = b_{8,46} b_{1,2} + b_{8,46} b_{1,1} + b_{6,29} b_{1,1}^3 + b_{4,14} b_{5,17} + b_{4,14}^2 b_{1,2} + b_{4,14}^2 b_{1,1} + b_{4,11} b_{5,20} + b_{4,11} b_{1,2}^2 b_{3,9} + b_{4,11} b_{1,2}^5 + b_{4,11} b_{4,14} b_{1,2} + b_{2,5} b_{1,2}^2 b_{5,20} + b_{2,5} b_{1,2}^4 b_{3,9} + b_{2,5}^2 b_{1,2}^2 b_{3,9} + b_{2,4} b_{1,1}^7 + b_{2,5}^2 b_{4,11} b_{1,2} + b_{2,4} b_{6,29} b_{1,0} + b_{2,4} b_{4,14} b_{3,8} + b_{2,4}^2 b_{4,14} b_{1,1} + b_{2,4}^4 b_{1,1} + b_{2,4}^4 b_{1,0}$

$Sq^2 c_{8,50} = b_{8,46} b_{1,1}^2 + b_{6,29} b_{1,1} b_{3,8} + b_{4,14} b_{1,2} b_{5,20} + b_{4,14} b_{1,1} b_{5,20} + b_{4,14} b_{1,1}^3 b_{3,8} + b_{4,14} b_{1,1}^6 + b_{4,11} b_{1,2} b_{5,20} + b_{4,11} b_{6,29} + b_{4,11} b_{4,14} b_{1,2}^2 + b_{2,5} b_{1,2} b_{7,38} + b_{2,5} b_{1,2}^3 b_{5,20} + b_{2,5} b_{8,46} + b_{2,5} b_{6,29} b_{1,0}^2 + b_{2,5} b_{4,14}^2 + b_{2,5} b_{4,11} b_{1,2} b_{3,9} + b_{2,5} b_{4,11}^2 + b_{2,5}^2 b_{1,0} b_{5,17} + b_{2,5}^2 b_{4,11} b_{1,2}^2 + b_{2,5}^3 b_{1,2} b_{3,9} + b_{2,5}^3 b_{4,14} + b_{2,4} b_{1,1}^8 + b_{2,4} b_{4,14} b_{1,1} b_{3,8} + b_{2,4} b_{4,14} b_{1,1}^4 + b_{2,4} b_{4,14}^2 + b_{2,4}^2 b_{1,1} b_{5,21} + b_{2,4}^3 b_{1,1} b_{3,8} + c_{8,50} b_{1,2}^2 + c_{8,50} b_{1,1} b_{1,2}$

$Sq^4 c_{8,50} = b_{4,14} b_{1,1}^8 + b_{4,14} b_{6,29} b_{1,1}^2 + b_{4,14}^2 b_{1,1} b_{3,9} + b_{4,14}^3 + b_{4,11} b_{4,14}^2 + b_{4,11}^2 b_{1,2} b_{3,9} + b_{4,11}^2 b_{4,14} + b_{4,11}^3 + b_{2,5} b_{4,14}^2 b_{1,2}^2 + b_{2,5} b_{4,11} b_{1,2}^3 b_{3,9} + b_{2,5} b_{4,11} b_{4,14} b_{1,2}^2 + b_{2,5} b_{4,11}^2 b_{1,2}^2 + b_{2,5}^2 b_{1,2}^3 b_{5,20} + b_{2,5}^2 b_{4,11}^2 + b_{2,5}^3 b_{4,11} b_{1,2}^2 + b_{2,5}^4 b_{4,14} + b_{2,5}^4 b_{4,11} + b_{2,4} b_{6,29} b_{1,1} b_{3,8} + b_{2,4} b_{4,14} b_{1,1}^6 + b_{2,4}^2 b_{6,29} b_{1,0}^2 + b_{2,4}^4 b_{4,14} + b_{2,4}^6 + c_{8,50} b_{1,2}^4 + c_{8,50} b_{1,1} b_{3,9} + c_{8,50} b_{1,0}^4 + b_{4,14} c_{8,50} + b_{4,11} c_{8,50} + b_{2,5} c_{8,50} b_{1,2}^2 + b_{2,5}^2 c_{8,50} + b_{2,4} c_{8,50} b_{1,1} b_{1,2} + b_{2,4} c_{8,50} b_{1,1}^2 + b_{2,4} c_{8,50} b_{1,0}^2 + b_{2,4}^2 c_{8,50}$

6. $SmallGroup(128, 934) \cong Syl_2(J_2)$

   Minimal generating set:

$$b_{1,0}, b_{1,1}, b_{1,2}, a_{2,4}, a_{2,5}, b_{3,8}, b_{3,9}, b_{4,14}, b_{5,20}, b_{5,21}, c_{8,49}, b_{10,83}.$$

$Sq^1 a_{2,4} = a_{2,4}b_{1,0}$

$Sq^1 a_{2,5} = a_{2,5}b_{1,0}$

$Sq^1 b_{3,8} = a_{2,4}^2$

$Sq^2 b_{3,8} = b_{5,21} + b_{1,2}^2 b_{3,9} + b_{1,1}b_{1,2}b_{3,9} + b_{1,1}b_{1,2}b_{3,8} + b_{1,1}^2 b_{3,8} + b_{4,14}b_{1,2} + a_{2,4}b_{3,9} + a_{2,4}a_{2,5}b_{1,0} + a_{2,4}^2 b_{1,0}$

$Sq^1 b_{3,9} = a_{2,4}a_{2,5} + a_{2,4}^2$

$Sq^2 b_{3,9} = b_{5,20} + b_{1,2}^2 b_{3,9} + b_{1,1}b_{1,2}b_{3,8} + b_{1,1}^2 b_{3,9} + b_{1,1}^2 b_{3,8} + b_{4,14}b_{1,2} + b_{4,14}b_{1,1} + a_{2,4}b_{1,0}^3 + a_{2,4}^2 b_{1,0}$

$Sq^1 b_{4,14} = b_{1,1}^2 b_{3,8} + a_{2,4}b_{1,0}^3$

$Sq^2 b_{4,14} = b_{1,1}b_{5,21} + b_{1,1}b_{1,2}^2 b_{3,9} + b_{1,1}b_{1,2}^2 b_{3,8} + b_{1,1}^2 b_{1,2}b_{3,9} + b_{4,14}b_{1,2}^2 + b_{4,14}b_{1,1}^2 + a_{2,4}b_{1,0}^4 + a_{2,4}a_{2,5}b_{1,0}^2 + a_{2,4}^2 b_{1,0}^2$

$Sq^1 b_{5,20} = b_{1,2}b_{5,20} + b_{1,1}^2 b_{1,2}b_{3,9} + a_{2,5}b_{4,14} + a_{2,4}a_{2,5}b_{1,0}^2 + a_{2,4}^2 b_{1,0}^2 + a_{2,4}^2 a_{2,5}$

$Sq^2 b_{5,20} = b_{1,2}^2 b_{5,20} + b_{1,1}^2 b_{5,20} + b_{1,1}^2 b_{1,2}^2 b_{3,8} + b_{1,1}^3 b_{1,2}b_{3,8} + b_{4,14}b_{1,1}b_{1,2}^2 + b_{4,14}b_{1,1}^2 b_{1,2}$

$Sq^4 b_{5,20} = b_{1,2}b_{3,8}b_{5,20} + b_{1,1}b_{3,8}b_{5,20} + b_{1,1}b_{1,2}^2 b_{3,8}b_{3,9} + b_{1,1}b_{1,2}^3 b_{5,20} + b_{1,1}^2 b_{1,2}b_{3,8}b_{3,9} + b_{1,1}^2 b_{1,2}^2 b_{5,20} + b_{1,1}^2 b_{1,2}^4 b_{3,9} + b_{1,1}^3 b_{1,2}^3 b_{3,9} + b_{1,1}^3 b_{1,2}^3 b_{3,8} + b_{1,1}^4 b_{5,20} + b_{1,1}^4 b_{1,2}^2 b_{3,8} + b_{1,1}^5 b_{1,2}b_{3,8} + b_{4,14}b_{5,20} + b_{4,14}b_{1,2}^2 b_{3,8} + b_{4,14}b_{1,2}^5 + b_{4,14}b_{1,1}b_{1,2}b_{3,9} + b_{4,14}b_{1,1}b_{1,2}b_{3,8} + b_{4,14}b_{1,1}b_{1,2}^4 + b_{4,14}b_{1,1}^2 b_{1,2}^3 + b_{4,14}b_{1,1}^3 b_{1,2}^2 + b_{4,14}^2 b_{1,2} + a_{2,4}b_{1,0}^7 + a_{2,4}^2 a_{2,5}b_{1,0}^3 + c_{8,49}b_{1,2}$

$Sq^1 b_{5,21} = b_{1,1}b_{5,21} + a_{2,4}b_{4,14} + a_{2,4}a_{2,5}b_{1,0}^2 + a_{2,4}^2 a_{2,5}$

$Sq^2 b_{5,21} = b_{1,2}^2 b_{5,20} + b_{1,1}b_{1,2}b_{5,20} + b_{1,1}b_{1,2}^3 b_{3,9} + b_{1,1}^2 b_{5,21} + b_{1,1}^2 b_{1,2}^2 b_{3,9} + b_{1,1}^2 b_{1,2}^2 b_{3,8} + b_{1,1}^3 b_{1,2}b_{3,8} + b_{4,14}b_{1,1}b_{1,2}^2 + b_{4,14}b_{1,1}^2 b_{1,2} + a_{2,4}b_{5,20} + a_{2,4}b_{4,14}b_{1,2} + a_{2,4}a_{2,5}b_{1,0}^3 + a_{2,4}^2 a_{2,5}b_{1,0}$

$Sq^4 b_{5,21} = b_{1,2}^4 b_{5,20} + b_{1,2}^6 b_{3,9} + b_{1,2}^6 b_{3,8} + b_{1,1}b_{3,8}b_{5,21} + b_{1,1}b_{3,8}b_{5,20} + b_{1,1}b_{1,2}^2 b_{3,8}b_{3,9} + b_{1,1}b_{1,2}^5 b_{3,8} + b_{1,1}^2 b_{1,2}^2 b_{5,20} + b_{1,1}^2 b_{1,2}^4 b_{3,9} + b_{1,1}^3 b_{1,2}^3 b_{3,8} + b_{1,1}^4 b_{5,21} + b_{1,1}^4 b_{5,20} + b_{1,1}^4 b_{1,2}^2 b_{3,8} + b_{1,1}^5 b_{1,2}b_{3,8} + b_{1,1}^6 b_{3,9} + b_{4,14}b_{5,21} + b_{4,14}b_{1,2}^2 b_{3,9} + b_{4,14}b_{1,2}^2 b_{3,8} + b_{4,14}b_{1,1}b_{1,2}^4 + b_{4,14}b_{1,1}^2 b_{3,8} + b_{4,14}b_{1,1}b_{1,2}b_{3,9} + b_{4,14}b_{1,1}^2 b_{1,2}^3 + b_{4,14}b_{1,1}^3 b_{1,2}^2 + b_{4,14}b_{1,1}^5 + b_{4,14}^2 b_{1,1} + a_{2,4}b_{4,14}b_{3,9} +$

$a_{2,4}^2a_{2,5}b_{1,0}^3 + c_{8,49}b_{1,1}$

$Sq^1c_{8,49} = b_{1,2}b_{3,8}b_{5,20} + b_{1,1}b_{1,2}^2b_{3,8}b_{3,9} + b_{1,1}b_{1,2}^3b_{5,20} + b_{1,1}^2b_{1,2}^2b_{5,20} + b_{1,1}^2b_{1,2}^4b_{3,9} + b_{1,1}^3b_{1,2}^3b_{3,9} + b_{1,1}^4b_{5,21} + b_{1,1}^4b_{5,20} + b_{1,1}^4b_{1,2}^2b_{3,8} + b_{1,1}^5b_{1,2}b_{3,8} + b_{1,1}^6b_{3,9} + b_{1,1}^6b_{3,8} + b_{4,14}b_{1,2}^2b_{3,8} + b_{4,14}b_{1,1}^2b_{3,8} + b_{4,14}b_{1,1}^2b_{1,2}^3 + b_{4,14}b_{1,1}^4b_{1,2} + a_{2,4}b_{4,14}b_{3,9}$

$Sq^2c_{8,49} = b_{1,1}^2b_{3,8}b_{5,21} + b_{1,1}^2b_{1,2}^2b_{3,8}b_{3,9} + b_{1,1}^2b_{1,2}^5b_{3,8} + b_{1,1}^3b_{1,2}b_{3,8}b_{3,9} + b_{1,1}^3b_{1,2}^2b_{5,20} + b_{1,1}^3b_{1,2}^4b_{3,8} + b_{1,1}^4b_{1,2}^3b_{3,9} + b_{1,1}^4b_{1,2}^3b_{3,8} + b_{1,1}^5b_{5,20} + b_{1,1}^5b_{1,2}^2b_{3,9} + b_{1,1}^6b_{1,2}b_{3,9} + b_{1,1}^7b_{3,9} + b_{1,1}^7b_{3,8} + b_{4,14}b_{1,2}b_{5,20} + b_{4,14}b_{1,2}^6 + b_{4,14}b_{1,1}b_{5,21} + b_{4,14}b_{1,1}b_{1,2}^5 + b_{4,14}b_{1,1}^2b_{1,2}b_{3,9} + b_{4,14}b_{1,1}^2b_{1,2}^4 + b_{4,14}b_{1,1}^4b_{1,2}^2 + b_{4,14}^2b_{1,2}^2 + a_{2,5}b_{4,14}^2 + a_{2,4}b_{4,14}^2 + a_{2,4}a_{2,5}b_{1,0}^6 + a_{2,4}^2b_{1,0}^6 + a_{2,4}^2a_{2,5}b_{1,0}^4 + c_{8,49}b_{1,1}b_{1,2}$

$Sq^4c_{8,49} = b_{1,1}b_{1,2}^3b_{3,8}b_{5,20} + b_{1,1}^2b_{1,2}^2b_{3,8}b_{5,20} + b_{1,1}^2b_{1,2}^4b_{3,8}b_{3,9} + b_{1,1}^3b_{1,2}b_{3,8}b_{5,20} + b_{1,1}^4b_{3,8}b_{5,21} + b_{1,1}^4b_{3,8}b_{5,20} + b_{1,1}^4b_{1,2}^2b_{3,8}b_{3,9} + b_{1,1}^4b_{1,2}^5b_{3,9} + b_{1,1}^5b_{1,2}b_{3,8}b_{3,9} + b_{1,1}^5b_{1,2}^2b_{5,20} + b_{1,1}^5b_{1,2}^4b_{3,9} + b_{1,1}^6b_{3,8}b_{3,9} + b_{1,1}^6b_{1,2}b_{5,20} + b_{1,1}^6b_{1,2}^3b_{3,9} + b_{1,1}^6b_{1,2}^3b_{3,8} + b_{4,14}b_{1,2}^3b_{5,20} + b_{1,1}^8b_{1,2}b_{3,9} + b_{1,1}^7b_{5,21} + b_{1,1}^7b_{5,20} + b_{1,1}^9b_{3,9} + b_{1,1}^9b_{3,8} + b_{4,14}b_{1,2}^5b_{3,8} + b_{4,14}b_{1,1}b_{1,2}^2b_{5,20} + b_{4,14}b_{1,1}b_{1,2}^4b_{3,9} + b_{4,14}b_{1,1}b_{1,2}^4b_{3,8} + b_{4,14}b_{1,1}^2b_{1,2}^3b_{3,9} + b_{4,14}b_{1,1}^2b_{1,2}^3b_{3,8} + b_{4,14}b_{1,1}^2b_{1,2}^6 + b_{4,14}b_{1,1}^3b_{5,21} + b_{4,14}b_{1,1}^4b_{1,2}b_{3,9} + b_{4,14}b_{1,1}^4b_{1,2}^4 + b_{4,14}b_{1,1}^5b_{3,9} + b_{4,14}b_{1,1}^5b_{1,2}^3 + b_{4,14}b_{1,1}^7b_{1,2} + b_{4,14}^2b_{1,2}b_{3,8} + b_{4,14}^2b_{1,2}^4 + b_{4,14}^2b_{1,1}b_{3,8} + b_{4,14}^2b_{1,1}b_{1,2}^3 + b_{4,14}^2b_{1,1}^2b_{1,2}^2 + b_{4,14}^2b_{1,1}^3b_{1,2} + b_{4,14}^3 + b_{4,14}^2b_{1,1}^4 + c_{8,49}b_{1,2}b_{3,8} + c_{8,49}b_{1,2}^4 + c_{8,49}b_{1,1}b_{3,8} + c_{8,49}b_{1,0}^4 + b_{4,14}c_{8,49} + a_{2,4}c_{8,49}b_{1,0}^2 + a_{2,4}a_{2,5}c_{8,49}$

$Sq^1b_{10,83} = b_{1,2}^3b_{3,9}b_{5,20} + b_{1,1}b_{1,2}^4b_{3,8}b_{3,9} + b_{1,1}b_{1,2}^5b_{5,20} + b_{1,1}^2b_{1,2}^3b_{3,8}b_{3,9} + b_{1,1}^2b_{1,2}^6b_{3,9} + b_{1,1}^3b_{3,8}b_{5,21} + b_{1,1}^3b_{1,2}^3b_{5,20} + b_{1,1}^3b_{1,2}^5b_{3,9} + b_{1,1}^4b_{1,2}b_{3,8}b_{3,9} + b_{1,1}^6b_{5,21} + b_{1,1}^6b_{5,20} + b_{1,1}^8b_{3,9} + b_{1,1}^6b_{1,2}^2b_{3,9} + b_{1,1}^8b_{3,8} + b_{4,14}b_{1,2}^4b_{3,8} + b_{4,14}b_{1,1}b_{1,2}^6 + b_{4,14}b_{1,1}^2b_{1,2}^2b_{3,9} + b_{4,14}b_{1,1}^2b_{1,2}^2b_{3,8} + b_{4,14}b_{1,1}^2b_{1,2}^5 + b_{4,14}b_{1,1}^3b_{1,2}^4 + b_{4,14}b_{1,1}^4b_{3,8} + b_{4,14}b_{1,1}^4b_{1,2}^3 + b_{4,14}b_{1,1}^5b_{1,2}^2 + a_{2,4}b_{4,14}b_{5,20} + a_{2,4}b_{4,14}^2b_{1,2} + a_{2,4}c_{8,49}b_{1,0}$

$Sq^2b_{10,83} = b_{1,2}^4b_{3,8}b_{5,20} + b_{1,2}^7b_{5,20} + b_{1,1}b_{1,2}^3b_{3,9}b_{5,20} + b_{1,1}b_{1,2}^5b_{3,8}b_{3,9} + b_{1,1}b_{1,2}^6b_{5,20} +$

$b_{1,1}b_{1,2}^8b_{3,9}+b_{1,1}b_{1,2}^8b_{3,8}+b_{1,1}^2b_{1,2}^2b_{3,9}b_{5,20}+b_{1,1}^2b_{1,2}^2b_{3,8}b_{5,20}+b_{1,1}^2b_{1,2}^4b_{3,8}b_{3,9}+b_{1,1}^2b_{1,2}^7b_{3,9}+b_{1,1}^2b_{1,2}^7b_{3,8}+b_{1,1}^3b_{1,2}b_{3,8}b_{5,20}+b_{1,1}^3b_{1,2}^3b_{3,8}b_{3,9}+b_{1,1}^3b_{1,2}^4b_{5,20}+b_{1,1}^4b_{3,9}b_{5,20}+b_{1,1}^4b_{3,8}b_{5,20}+b_{1,1}^4b_{1,2}^2b_{3,8}b_{3,9}+b_{1,1}^5b_{1,2}b_{3,8}b_{3,9}+b_{1,1}^5b_{1,2}^4b_{3,8}+b_{1,1}^6b_{3,8}b_{3,9}+b_{1,1}^6b_{1,2}^3b_{3,9}+b_{1,1}^7b_{1,2}^2b_{3,9}+b_{1,1}^7b_{1,2}^2b_{3,8}+b_{1,1}^8b_{1,2}b_{3,9}+b_{1,1}^8b_{1,2}b_{3,8}+b_{1,1}^9b_{3,9}+b_{1,1}^9b_{3,8}+b_{10,83}b_{1,2}^2+b_{10,83}b_{1,1}b_{1,2}+b_{10,83}b_{1,1}^2+b_{4,14}b_{3,9}b_{5,20}+b_{4,14}b_{3,8}b_{5,21}+b_{4,14}b_{3,8}b_{5,20}+b_{4,14}b_{1,2}^2b_{3,8}b_{3,9}+b_{4,14}b_{1,2}^3b_{5,20}+b_{4,14}b_{1,1}b_{1,2}b_{3,8}b_{3,9}+b_{4,14}b_{1,1}b_{1,2}^2b_{5,20}+b_{4,14}b_{1,1}b_{1,2}^4b_{3,9}+b_{4,14}b_{1,1}b_{1,2}^4b_{3,8}+b_{4,14}b_{1,1}^3b_{5,21}+b_{4,14}b_{1,1}^2b_{3,8}b_{3,9}+b_{4,14}b_{1,1}^2b_{1,2}b_{5,20}+b_{4,14}b_{1,1}^2b_{1,2}^3b_{3,8}+b_{4,14}b_{1,1}^2b_{1,2}^6+b_{4,14}b_{1,1}^3b_{1,2}^2b_{3,9}+b_{4,14}b_{1,1}^3b_{1,2}^2b_{3,8}+b_{4,14}b_{1,1}^4b_{1,2}^4+b_{4,14}b_{1,1}^5b_{3,9}+b_{4,14}b_{1,1}^5b_{3,8}+b_{4,14}b_{1,1}^5b_{1,2}^3+b_{4,14}b_{1,1}^6b_{1,2}^2+b_{4,14}b_{1,1}^7b_{1,2}+b_{4,14}^2b_{1,2}b_{3,9}+b_{4,14}^2b_{1,1}b_{3,9}+b_{4,14}^2b_{1,1}b_{3,8}+b_{4,14}^2b_{1,1}b_{1,2}^3+b_{4,14}^2b_{1,1}^3b_{1,2}+a_{2,4}a_{2,5}b_{1,0}^8+a_{2,4}^2b_{1,0}^8+a_{2,4}^2a_{2,5}b_{1,0}^6+c_{8,49}b_{1,2}b_{3,8}+c_{8,49}b_{1,2}^4+c_{8,49}b_{1,1}b_{3,9}+c_{8,49}b_{1,1}^2b_{1,2}^2+c_{8,49}b_{1,1}^3b_{1,2}+c_{8,49}b_{1,0}^4+a_{2,4}^2c_{8,49}+b_{4,14}b_{1,1}^4b_{1,2}b_{3,9}$

$Sq^4b_{10,83} = b_{4,14}b_{1,1}b_{1,2}^3b_{3,8}b_{3,9}+b_{1,2}^6b_{3,8}b_{5,20}+b_{1,1}b_{1,2}^7b_{3,8}b_{3,9}+b_{1,1}^2b_{1,2}^4b_{3,9}b_{5,20}+b_{1,1}^2b_{1,2}^4b_{3,8}b_{5,20}+b_{1,1}^2b_{1,2}^7b_{5,20}+b_{1,1}^2b_{1,2}^9b_{3,9}+b_{1,1}^2b_{1,2}^9b_{3,8}+b_{1,1}^3b_{1,2}^6b_{5,20}+b_{1,1}^3b_{1,2}^8b_{3,9}+b_{1,1}^4b_{1,2}^2b_{3,8}b_{5,20}+b_{1,1}^4b_{1,2}^5b_{5,20}+b_{1,1}^5b_{1,2}b_{3,8}b_{5,20}+b_{1,1}^5b_{1,2}^6b_{3,8}+b_{1,1}^6b_{3,9}b_{5,21}+b_{1,1}^6b_{3,9}b_{5,20}+b_{1,1}^6b_{3,8}b_{5,21}+b_{1,1}^6b_{1,2}^2b_{3,8}b_{3,9}+b_{1,1}^6b_{1,2}^3b_{5,20}+b_{1,1}^6b_{1,2}^5b_{3,9}+b_{1,1}^6b_{1,2}^5b_{3,8}+b_{1,1}^7b_{1,2}b_{3,8}b_{3,9}+b_{1,1}^7b_{1,2}^2b_{5,20}+b_{1,1}^7b_{1,2}^4b_{3,9}+b_{1,1}^9b_{5,21}+b_{1,1}^9b_{5,20}+b_{1,1}^9b_{1,2}^2b_{3,8}+b_{1,1}^{10}b_{1,2}b_{3,8}+b_{1,1}^{11}b_{3,9}+b_{10,83}b_{1,2}b_{3,9}+b_{10,83}b_{1,1}b_{3,8}+b_{10,83}b_{1,1}^3b_{1,2}+b_{4,14}b_{1,2}^2b_{3,8}b_{5,20}+b_{4,14}b_{1,1}b_{1,2}b_{3,8}b_{5,20}+b_{4,14}b_{1,1}b_{1,2}^6b_{3,9}+b_{4,14}b_{1,1}b_{1,2}^6b_{3,8}+b_{4,14}b_{1,1}^2b_{3,9}b_{5,21}+b_{4,14}b_{1,2}^7b_{3,8}+b_{4,14}b_{1,1}^2b_{3,9}b_{5,20}+b_{4,14}b_{1,2}^{10}+b_{4,14}b_{1,1}^2b_{1,2}^3b_{5,20}+b_{4,14}b_{1,1}^2b_{1,2}^5b_{3,9}+b_{4,14}b_{1,1}^3b_{1,2}b_{3,8}b_{3,9}+b_{4,14}b_{1,1}^3b_{1,2}^2b_{5,20}+b_{4,14}b_{1,1}^3b_{1,2}^4b_{3,8}+b_{4,14}b_{1,1}^4b_{3,8}b_{3,9}+b_{4,14}b_{1,1}^4b_{1,2}b_{5,20}+b_{4,14}b_{1,1}^4b_{1,2}^3b_{3,9}+b_{4,14}b_{1,1}^4b_{1,2}^6+b_{4,14}b_{1,1}^5b_{5,20}+b_{4,14}b_{1,1}^5b_{1,2}^5+b_{4,14}b_{1,1}^6b_{1,2}^4+b_{4,14}b_{1,1}^7b_{3,9}+b_{4,14}b_{1,1}^8b_{1,2}^2+b_{4,14}b_{1,1}^{10}+b_{4,14}b_{10,83}+b_{4,14}^2b_{3,8}b_{3,9}+b_{4,14}^2b_{1,2}^6+b_{4,14}^2b_{1,1}b_{5,21}+b_{4,14}^2b_{1,1}b_{1,2}^2b_{3,8}+b_{4,14}^2b_{1,1}^2b_{1,2}b_{3,9}+b_{4,14}^2b_{1,1}^3b_{3,8}+b_{4,14}^2b_{1,1}^3b_{1,2}^3+b_{4,14}^2b_{1,1}^5b_{1,2}+b_{4,14}^2b_{1,1}^6+b_{4,14}^3b_{1,2}^2+a_{2,5}b_{4,14}^3+a_{2,4}b_{4,14}^3+a_{2,4}a_{2,5}b_{1,0}^{10}+a_{2,4}^2b_{1,0}^{10}+a_{2,4}^2a_{2,5}b_{1,0}^8+c_{8,49}b_{3,8}b_{3,9}+c_{8,49}b_{1,2}^3b_{3,9}+c_{8,49}b_{1,2}^3b_{3,8}+c_{8,49}b_{1,2}^6+$

$c_{8,49}b_{1,1}b_{5,20}+c_{8,49}b_{1,1}b_{1,2}^5+c_{8,49}b_{1,1}^2b_{1,2}b_{3,9}+c_{8,49}b_{1,1}^3b_{3,8}+c_{8,49}b_{1,1}^3b_{1,2}^3+c_{8,49}b_{1,1}^5b_{1,2}+c_{8,49}b_{1,1}^6+c_{8,49}b_{1,0}^6+b_{4,14}c_{8,49}b_{1,1}b_{1,2}+a_{2,5}b_{4,14}c_{8,49}+a_{2,4}b_{4,14}c_{8,49}+a_{2,4}^2a_{2,5}c_{8,49}$

$Sq^8b_{10,83} = b_{1,1}^2b_{1,2}^{13}b_{3,9}+b_{1,1}^2b_{1,2}^{13}b_{3,8}+b_{1,1}^3b_{1,2}^{12}b_{3,8}+b_{1,1}^4b_{1,2}^{11}b_{3,9}+b_{1,1}^5b_{1,2}^7b_{3,8}b_{3,9}+b_{1,1}^6b_{1,2}^9b_{3,9}+b_{1,1}^6b_{1,2}^9b_{3,8}+b_{1,1}^7b_{1,2}^6b_{5,20}+b_{1,1}^8b_{1,2}^2b_{3,9}b_{5,20}+b_{1,1}^8b_{1,2}^5b_{5,20}+b_{1,1}^9b_{1,2}^3b_{3,8}b_{3,9}+b_{1,1}^{10}b_{3,8}b_{5,20}+b_{1,1}^{10}b_{1,2}^2b_{3,8}b_{3,9}+b_{1,1}^{10}b_{1,2}^3b_{5,20}+b_{1,1}^{10}b_{1,2}^5b_{3,9}+b_{1,1}^{11}b_{1,2}^2b_{5,20}+b_{1,1}^{11}b_{1,2}^4b_{3,8}+b_{1,1}^{12}b_{1,2}b_{5,20}+b_{10,83}b_{1,1}b_{1,2}b_{3,8}b_{3,9}+b_{1,1}^{12}b_{1,2}^3b_{3,8}+b_{1,1}^{13}b_{1,2}^2b_{3,9}+b_{1,1}^{13}b_{1,2}^2b_{3,8}+b_{1,1}^{14}b_{1,2}b_{3,9}+b_{1,1}^{14}b_{1,2}b_{3,8}+b_{1,1}^{15}b_{3,9}+b_{10,83}b_{3,9}b_{5,21}+b_{10,83}b_{3,9}b_{5,20}+b_{10,83}b_{3,8}b_{5,21}+b_{10,83}b_{1,2}^5b_{3,9}+b_{10,83}b_{1,1}b_{1,2}^2b_{5,20}+b_{10,83}b_{1,1}b_{1,2}^7+b_{10,83}b_{1,1}^2b_{1,2}b_{5,20}+b_{10,83}b_{1,1}^3b_{5,21}+b_{10,83}b_{1,1}^3b_{5,20}+b_{10,83}b_{1,2}^8+b_{10,83}b_{1,1}^4b_{1,2}b_{3,9}+b_{10,83}b_{1,1}^4b_{1,2}^4+b_{10,83}b_{1,1}^5b_{3,9}+b_{10,83}b_{1,1}^5b_{3,8}+b_{10,83}b_{1,1}^5b_{1,2}^3+b_{10,83}b_{1,1}^8+b_{4,14}b_{1,2}^8b_{3,8}b_{3,9}+b_{4,14}b_{1,2}^{11}b_{3,8}+b_{4,14}b_{1,1}b_{1,2}^7b_{3,8}b_{3,9}+b_{4,14}b_{1,1}b_{1,2}^{10}b_{3,9}+b_{4,14}b_{1,1}^2b_{1,2}^4b_{3,9}b_{5,20}+b_{4,14}b_{1,1}^2b_{1,2}^4b_{3,8}b_{5,20}+b_{4,14}b_{1,1}^2b_{1,2}^6b_{3,8}b_{3,9}+b_{4,14}b_{1,1}^3b_{1,2}^3b_{3,9}b_{5,20}+b_{4,14}b_{1,1}^2b_{1,2}^{12}+b_{4,14}b_{1,1}^3b_{1,2}^3b_{3,8}b_{5,20}+b_{4,14}b_{1,1}^3b_{1,2}^8b_{3,9}+b_{4,14}b_{1,1}^3b_{1,2}^{11}+b_{4,14}b_{1,1}^4b_{1,2}^2b_{3,8}b_{5,20}+b_{4,14}b_{1,1}^4b_{1,2}^4b_{3,8}b_{3,9}+b_{4,14}b_{1,1}^4b_{1,2}^5b_{5,20}+b_{4,14}b_{1,1}^5b_{1,2}^4b_{5,20}+b_{4,14}b_{1,1}^6b_{3,8}b_{5,20}+a_{2,4}c_{8,49}^2+b_{4,14}b_{1,1}^6b_{1,2}^5b_{3,9}+b_{4,14}b_{1,1}^6b_{1,2}^5b_{3,8}+b_{4,14}b_{1,1}^6b_{1,2}^8+b_{4,14}b_{1,1}^8b_{1,2}^3b_{3,9}+b_{4,14}b_{1,1}^9b_{5,20}+b_{4,14}b_{1,1}^9b_{1,2}^2b_{3,8}+b_{4,14}b_{1,1}^{10}b_{1,2}b_{3,9}+b_{4,14}b_{1,1}^{10}b_{1,2}b_{3,8}+b_{4,14}b_{1,1}^{12}b_{1,2}^2+b_{4,14}b_{10,83}b_{1,2}b_{3,8}+b_{4,14}b_{10,83}b_{1,2}^4+b_{4,14}b_{10,83}b_{1,1}b_{3,8}+b_{4,14}b_{10,83}b_{1,1}b_{1,2}^3+b_{4,14}^2b_{1,2}^2b_{3,9}b_{5,20}+b_{4,14}b_{1,1}^{13}b_{1,2}+b_{4,14}b_{1,1}^{14}+b_{4,14}^2b_{1,2}^2b_{3,8}b_{5,20}+b_{4,14}^2b_{1,2}^5b_{5,20}+b_{4,14}^2b_{1,2}^7b_{3,9}+b_{4,14}^2b_{1,2}^7b_{3,8}+b_{4,14}^2b_{1,1}b_{1,2}^4b_{5,20}+b_{4,14}^2b_{1,1}b_{1,2}^6b_{3,9}+b_{4,14}^2b_{1,1}b_{1,2}^6b_{3,8}+b_{4,14}^2b_{1,1}^2b_{3,8}b_{5,21}+b_{4,14}^2b_{1,1}^2b_{3,8}b_{5,20}+b_{4,14}^2b_{1,1}^2b_{1,2}^8+b_{4,14}^2b_{1,1}^3b_{1,2}^2b_{5,20}+b_{4,14}^2b_{1,1}^3b_{1,2}^4b_{3,9}+b_{4,14}^2b_{1,1}^4b_{3,8}b_{3,9}+b_{4,14}^2b_{1,1}^4b_{1,2}b_{5,20}+b_{4,14}^2b_{1,1}^4b_{1,2}^3b_{3,9}+b_{4,14}^2b_{1,1}^7b_{3,9}+b_{4,14}^2b_{1,1}^7b_{1,2}^3+b_{4,14}^2b_{1,1}^8b_{1,2}^2+b_{4,14}^2b_{1,1}^{10}+b_{4,14}^3b_{1,2}b_{5,20}+b_{4,14}^3b_{1,1}b_{1,2}^2b_{3,9}+b_{4,14}^3b_{1,1}b_{1,2}^2b_{3,8}+b_{4,14}^3b_{1,1}^2b_{1,2}b_{3,9}+b_{4,14}^3b_{1,1}^2b_{1,2}^4+b_{4,14}^3b_{1,1}^3b_{3,9}+b_{4,14}^3b_{1,1}^3b_{3,8}+b_{4,14}^4b_{1,1}^2+a_{2,4}b_{4,14}^4+a_{2,4}^2b_{1,0}^{14}+c_{8,49}b_{1,2}^5b_{5,20}+c_{8,49}b_{1,2}^7b_{3,9}+c_{8,49}b_{1,1}b_{1,2}^3b_{3,8}b_{3,9}+c_{8,49}b_{1,1}b_{1,2}^4b_{5,20}+c_{8,49}b_{1,1}^2b_{3,9}b_{5,21}+c_{8,49}b_{1,1}^2b_{1,2}^8+c_{8,49}b_{1,1}^3b_{1,2}^2b_{5,20}+c_{8,49}b_{1,1}^4b_{3,8}b_{3,9}+c_{8,49}b_{1,1}^4b_{1,2}^3b_{3,9}+c_{8,49}b_{1,1}^4b_{1,2}^3b_{3,8}+c_{8,49}b_{1,1}^4b_{1,2}^6+c_{8,49}b_{1,1}^5b_{5,21}+c_{8,49}b_{1,1}^5b_{1,2}^2b_{3,8}+c_{8,49}b_{1,1}^8b_{1,2}^2+c_{8,49}b_{1,1}^{10}+$

$c_{8,49}b_{1,1}^9b_{1,2} + b_{4,14}c_{8,49}b_{1,2}b_{5,20} + b_{4,14}c_{8,49}b_{1,2}^6 + b_{4,14}c_{8,49}b_{1,1}b_{5,21} + b_{4,14}c_{8,49}b_{1,1}b_{5,20} + b_{4,14}c_{8,49}b_{1,1}b_{1,2}^2b_{3,8} + b_{4,14}^2b_{1,1}^2b_{1,2}^2b_{3,8}b_{3,9} + b_{4,14}c_{8,49}b_{1,1}b_{1,2}^5 + b_{4,14}c_{8,49}b_{1,1}^2b_{1,2}b_{3,8} + b_{4,14}c_{8,49}b_{1,1}^2b_{1,2}^4 + b_{4,14}c_{8,49}b_{1,1}^3b_{3,9} + b_{4,14}^2c_{8,49}b_{1,2}^2 + a_{2,4}^2a_{2,5}c_{8,49}b_{1,0}^4 + c_{8,49}^2b_{1,1}b_{1,2} + c_{8,49}^2b_{1,1}^2 + c_{8,49}^2b_{1,0}^2 + b_{4,14}^2b_{1,1}b_{1,2}^3b_{3,8}b_{3,9} + b_{4,14}b_{1,1}^6b_{1,2}^2b_{3,8}b_{3,9} + c_{8,49}b_{1,1}b_{1,2}b_{3,9}b_{5,20} + c_{8,49}b_{1,1}b_{1,2}b_{3,8}b_{5,20}$

5.2 Steenrod operations for non-prime power groups

Theoretically, the Steenrod operations on the cohomology rings of a non-prime group are known if we have already known those on the cohomology ring of the Sylow subgroup of the given group. Thus we could compute the Steenrod squares on the cohomology rings of several non-prime groups whose Sylow 2-subgroup is of order less than or equal 128 and the Steenrod squares are computed. From the first run of our program, we have the following results for some special non-prime groups.

5.2.1 Alternating groups

1. A_8 and A_9:

   $Syl_2(A_8) \cong Syl_2(A_9) \cong SmallGroup(64, 138)$.

   In Green and King's package these groups have the same cohomology ring structure. Minimal generating set:

   $$b_{2,0}, b_{3,1}, b_{3,0}, c_{4,0}, b_{5,0}, b_{6,2}, b_{6,0}, b_{7,6}, b_{7,4}.$$

   And our computations produce the same Steenrod operations:

   $Sq^1b_{2,0} = b_{3,1} + b_{3,0}$

   $Sq^1b_{3,0} = 0$

$$Sq^2 b_{3,0} = b_{2,0} b_{3,0}$$

$$Sq^1 b_{3,1} = 0$$

$$Sq^2 b_{3,1} = b_{5,0} + b_{2,0} b_{3,1}$$

$$Sq^1 c_{4,0} = b_{5,0}$$

$$Sq^2 c_{4,0} = b_{6,0} + b_{2,0} c_{4,0}$$

$$Sq^1 b_{5,0} = 0$$

$$Sq^2 b_{5,0} = b_{2,0} b_{5,0}$$

$$Sq^4 b_{5,0} = b_{6,0} b_{3,1} + c_{4,0} b_{5,0}$$

$$Sq^1 b_{6,0} = b_{7,4}$$

$$Sq^2 b_{6,0} = b_{2,0} b_{6,0}$$

$$Sq^4 b_{6,0} = c_{4,0} b_{6,0}$$

$$Sq^1 b_{6,2} = b_{7,6}$$

$$Sq^2 b_{6,2} = b_{3,1} b_{5,0} + b_{2,0} b_{3,1}^2 + b_{2,0} b_{3,0}^2 + b_{2,0} b_{6,2} + b_{2,0}^2 c_{4,0}$$

$$Sq^4 b_{6,2} = b_{2,0} b_{3,1} b_{5,0} + b_{2,0}^2 b_{3,1}^2 + b_{2,0}^2 b_{3,0}^2 + b_{2,0}^2 b_{6,0} + c_{4,0} b_{6,2} + b_{2,0}^3 c_{4,0}$$

$$Sq^1 b_{7,4} = 0$$

$$Sq^2 b_{7,4} = 0$$

$$Sq^4 b_{7,4} = c_{4,0} b_{7,4}$$

$$Sq^1 b_{7,6} = 0$$

$$Sq^2 b_{7,6} = 0$$

$$Sq^4 b_{7,6} = c_{4,0} b_{7,6} + b_{2,0}^2 c_{4,0} b_{3,0}$$

5.2.2 Symmetric groups

1. $\mathcal{S}_8$ and $\mathcal{S}_9$:

   $Syl_2(\mathcal{S}_8) \cong Syl_2(\mathcal{S}_9) \cong SmallGroup(128, 928).$

In Green and King's package these groups have the same cohomology ring structure. Minimal generating set:

$$b_{1,0}, b_{2,0}, b_{3,0}, b_{3,1}, c_{4,0}, b_{5,8}, b_{6,0}, b_{7,17}.$$

And our computations produce the same Steenrod operations:

$Sq^1 b_{2,0} = b_{3,1} + b_{3,0} + b_{2,0}b_{1,0}$

$Sq^1 b_{3,0} = b_{1,0}b_{3,1} + b_{1,0}b_{3,0}$

$Sq^2 b_{3,0} = b_{5,8} + b_{1,0}^2 b_{3,1} + b_{1,0}^2 b_{3,0} + b_{2,0}b_{3,0} + c_{4,0}b_{1,0}$

$Sq^1 b_{3,1} = 0$

$Sq^2 b_{3,1} = b_{1,0}^2 b_{3,1} + b_{1,0}^2 b_{3,0} + b_{2,0}b_{3,1} + c_{4,0}b_{1,0}$

$Sq^1 c_{4,0} = b_{5,8} + b_{1,0}^2 b_{3,1}$

$Sq^2 c_{4,0} = b_{1,0}^3 b_{3,0} + b_{6,0} + b_{2,0}c_{4,0}$

$Sq^1 b_{5,8} = 0$

$Sq^2 b_{5,8} = b_{1,0}^2 b_{5,8} + b_{2,0}b_{5,8}$

$Sq^4 b_{5,8} = b_{1,0}^3 b_{3,0}^2 + b_{1,0}^4 b_{5,8} + b_{1,0}^6 b_{3,0} + b_{6,0}b_{3,0} + b_{2,0}b_{1,0}^2 b_{5,8} + b_{2,0}b_{1,0}^4 b_{3,1} + c_{4,0}b_{5,8} +$
$c_{4,0}b_{1,0}^2 b_{3,1} + c_{4,0}b_{1,0}^2 b_{3,0} + c_{4,0}b_{1,0}^5$

$Sq^1 b_{6,0} = b_{7,17}$

$Sq^2 b_{6,0} = b_{2,0}b_{6,0}$

$Sq^4 b_{6,0} = c_{4,0}b_{6,0}$

$Sq^1 b_{7,17} = 0$

$Sq^2 b_{7,17} = 0$

$Sq^4 b_{7,17} = c_{4,0}b_{7,17}$

5.2.3 Mathieu groups

1. M_9:

   $Syl_2(M_9) \cong SmallGroup(8, 4)$.

   Minimal generating set: $a_{1,0}, a_{1,1}, c_{4,0}$.

   $Sq^1 c_{4,0} = 0$

   $Sq^2 c_{4,0} = 0$

2. M_{10}:

   $Syl_2(M_{10}) \cong SmallGroup(16, 8)$.

   Minimal generating set: $a_{1,0}, b_{3,0}, c_{4,0}, b_{5,0}$.

   $Sq^1 b_{3,0} = 0$

   $Sq^2 b_{3,0} = b_{5,0}$

   $Sq^1 c_{4,0} = 0$

   $Sq^2 c_{4,0} = b_{3,0}^2$

   $Sq^1 b_{5,0} = b_{3,0}^2$

   $Sq^2 b_{5,0} = 0$

   $Sq^4 b_{5,0} = b_{3,0}^3 + c_{4,0} b_{5,0}$

3. M_{11}:

   $Syl_2(M_{11}) \cong SmallGroup(16, 8)$.

   Minimal generating set: $b_{3,0}, c_{4,0}, b_{5,0}$.

   $Sq^1 b_{3,0} = 0$

   $Sq^2 b_{3,0} = b_{5,0}$

   $Sq^1 c_{4,0} = 0$

   $Sq^2 c_{4,0} = b_{3,0}^2$

$Sq^1 b_{5,0} = b_{3,0}^2$

$Sq^2 b_{5,0} = 0$

$Sq^4 b_{5,0} = b_{3,0}^3 + c_{4,0} b_{5,0}$

4. M_{12}:

   $Syl_2(M_{12}) \cong SmallGroup(64, 134)$.

   Minimal generating set: $b_{2,0}, b_{3,0}, b_{3,1}, b_{3,2}, c_{4,0}, b_{5,0}, b_{6,3}, b_{7,1}$.

   $Sq^1 b_{2,0} = b_{3,0}$

   $Sq^1 b_{3,0} = 0$

   $Sq^2 b_{3,0} = b_{2,0} b_{3,0}$

   $Sq^1 b_{3,1} = b_{2,0}^2$

   $Sq^2 b_{3,1} = b_{2,0} b_{3,1} + b_{2,0} b_{3,0}$

   $Sq^1 b_{3,2} = 0$

   $Sq^2 b_{3,2} = b_{5,0}$

   $Sq^1 c_{4,0} = 0$

   $Sq^2 c_{4,0} = b_{3,2}^2 + b_{6,3} + b_{2,0} c_{4,0}$

   $Sq^1 b_{5,0} = b_{3,2}^2$

   $Sq^2 b_{5,0} = 0$

   $Sq^4 b_{5,0} = b_{3,2}^3 + b_{6,3} b_{3,2} + c_{4,0} b_{5,0}$

   $Sq^1 b_{6,3} = b_{7,1} + c_{4,0} b_{3,2}$

   $Sq^2 b_{6,3} = b_{2,0} b_{6,3}$

   $Sq^4 b_{6,3} = b_{3,2} b_{7,1} + c_{4,0} b_{3,2}^2 + c_{4,0} b_{6,3}$

   $Sq^1 b_{7,1} = 0$

   $Sq^2 b_{7,1} = b_{3,2}^3 + b_{6,3} b_{3,2} + c_{4,0} b_{5,0}$

   $Sq^4 b_{7,1} = b_{3,2}^2 b_{5,0} + b_{6,3} b_{5,0} + b_{2,0}^2 b_{7,1} + c_{4,0} b_{7,1}$

5. M_{21}:

$Syl_2(M_{21}) \cong SmallGroup(64, 242)$.

Minimal generating set:

$a_{2,0}, a_{2,1}, b_{3,0}, b_{3,1}, b_{5,0}, b_{5,1}, b_{5,2}, b_{5,3}, b_{6,0}, b_{6,3}, b_{6,5}, c_{8,2}, b_{9,0}, b_{9,1}, b_{9,2}, b_{9,3}, c_{12,9}, c_{12,13}$.

$Sq^1 a_{2,0} = 0$

$Sq^1 a_{2,1} = 0$

$Sq^1 b_{3,0} = 0$

$Sq^2 b_{3,0} = b_{5,3} + b_{5,2} + b_{5,1} + a_{2,0}b_{3,0}$

$Sq^1 b_{3,1} = 0$

$Sq^2 b_{3,1} = b_{5,2} + b_{5,0} + a_{2,0}b_{3,0}$

$Sq^1 b_{5,0} = b_{3,1}^2 + b_{3,0}^2 + b_{6,5} + b_{6,4}$

$Sq^2 b_{5,0} = a_{2,0}b_{5,0}$

$Sq^4 b_{5,0} = b_{9,0} + b_{3,1}^3 + b_{3,0}^2 b_{3,1} + b_{6,5}b_{3,0} + b_{6,4}b_{3,1} + b_{6,0}b_{3,1}$

$Sq^1 b_{5,1} = b_{6,5}$

$Sq^2 b_{5,1} = 0$

$Sq^4 b_{5,1} = b_{9,1} + b_{6,0}b_{3,1}$

$Sq^1 b_{5,2} = b_{3,0}^2 + b_{6,5} + b_{6,4}$

$Sq^2 b_{5,2} = 0$

$Sq^4 b_{5,2} = b_{9,3} + b_{9,2} + b_{6,5}b_{3,0} + b_{6,4}b_{3,1} + b_{6,4}b_{3,0} + b_{6,0}b_{3,1}$

$Sq^1 b_{5,3} = b_{6,4}$

$Sq^2 b_{5,3} = a_{2,0}b_{5,0}$

$Sq^4 b_{5,3} = b_{9,2} + b_{3,0}^2 b_{3,1} + b_{3,0}^3 + b_{6,4}b_{3,0}$

$Sq^1 b_{6,0} = a_{2,0}b_{5,0}$

$Sq^2 b_{6,0} = b_{3,1}b_{5,1} + b_{3,0}b_{5,0}$

$Sq^4 b_{6,0} = b_{5,0} b_{5,1}$

$Sq^1 b_{6,4} = 0$

$Sq^2 b_{6,4} = 0$

$Sq^4 b_{6,4} = b_{5,3}^2$

$Sq^1 b_{6,5} = 0$

$Sq^2 b_{6,5} = 0$

$Sq^4 b_{6,5} = b_{5,1}^2$

$Sq^1 c_{8,6} = b_{3,0}^3 + b_{6,4} b_{3,0}$

$Sq^2 c_{8,6} = b_{5,2} b_{5,3} + b_{5,2}^2 + b_{5,1}^2 + a_{2,0} c_{8,6}$

$Sq^4 c_{8,6} = b_{3,1} b_{9,3} + b_{3,1} b_{9,1} + b_{3,1} b_{9,0} + b_{3,1}^4 + b_{3,0} b_{9,3} + b_{3,0} b_{9,2} + b_{3,0} b_{9,1} + b_{3,0} b_{9,0} +$

$b_{6,4} b_{3,0}^2 + b_{6,4}^2 + b_{6,0} b_{3,1}^2 + b_{6,0} b_{6,5} + b_{6,0}^2 + c_{12,10}$

$Sq^1 b_{9,0} = b_{5,0}^2$

$Sq^2 b_{9,0} = b_{3,1}^2 b_{5,1} + b_{3,1}^2 b_{5,0} + b_{6,5} b_{5,1} + b_{6,5} b_{5,0} + a_{2,0} b_{9,1} + a_{2,0} b_{9,0}$

$Sq^4 b_{9,0} = b_{3,1} b_{5,1}^2 + b_{3,1} b_{5,0}^2 + b_{3,0} b_{5,1}^2 + b_{3,0} b_{5,0}^2$

$Sq^8 b_{9,0} = b_{3,1}^4 b_{5,1} + b_{3,0} b_{5,0} b_{9,1} + b_{3,0} b_{5,0} b_{9,0} + b_{6,5}^2 b_{5,0} + b_{6,0} b_{3,1}^2 b_{5,1} + b_{6,0} b_{3,1}^2 b_{5,0} +$

$b_{6,0}^2 b_{5,1} + b_{6,0}^2 b_{5,0} + c_{12,14} b_{5,1} + c_{12,14} b_{5,0} + c_{12,10} b_{5,1} + c_{12,10} b_{5,0} + c_{8,6} b_{9,1} + c_{8,6} b_{3,1}^3 +$

$c_{8,6} b_{3,0}^2 b_{3,1} + b_{6,5} c_{8,6} b_{3,0} + b_{6,4} c_{8,6} b_{3,1} + b_{6,0} c_{8,6} b_{3,0} + a_{2,0} c_{12,14} b_{3,0} + a_{2,0} c_{12,10} b_{3,0}$

$Sq^1 b_{9,1} = b_{5,1}^2$

$Sq^2 b_{9,1} = b_{3,1}^2 b_{5,1} + a_{2,0} b_{9,0}$

$Sq^4 b_{9,1} = b_{3,0} b_{5,0}^2$

$Sq^8 b_{9,1} = b_{3,0} b_{5,0} b_{9,1} + b_{6,0} b_{6,5} b_{5,1} + b_{6,0} b_{6,5} b_{5,0} + b_{6,0}^2 b_{5,0} + c_{12,14} b_{5,1} + c_{12,14} b_{5,0} +$

$c_{8,6} b_{9,0} + c_{8,6} b_{3,1}^3 + c_{8,6} b_{3,0}^2 b_{3,1} + b_{6,5} c_{8,6} b_{3,0} + b_{6,4} c_{8,6} b_{3,1} + b_{6,0} c_{8,6} b_{3,0} + a_{2,0} c_{12,14} b_{3,0}$

$Sq^1 b_{9,2} = b_{5,3}^2$

$Sq^2 b_{9,2} = b_{3,0}^2 b_{5,3} + b_{6,5} b_{5,1} + b_{6,5} b_{5,0} + b_{6,4} b_{5,3} + b_{6,4} b_{5,2} + a_{2,0} b_{9,0}$

$Sq^4 b_{9,2} = b_{3,1}b_{5,3}^2 + b_{3,1}b_{5,2}^2 + b_{3,1}b_{5,1}^2 + b_{3,0}b_{5,2}^2 + b_{3,0}b_{5,1}^2$

$Sq^8 b_{9,2} = b_{3,0}b_{5,2}b_{9,3} + b_{3,0}^3 b_{3,1}b_{5,2} + b_{6,5}^2 b_{5,1} + b_{6,5}^2 b_{5,0} + b_{6,4}b_{3,0}b_{3,1}b_{5,2} + b_{6,4}b_{3,0}^2 b_{5,3} + b_{6,4}b_{3,0}^2 b_{5,2} + b_{6,4}^2 b_{5,2} + c_{12,14}b_{5,3} + c_{12,10}b_{5,3} + c_{12,10}b_{5,2} + b_{6,4}c_{8,6}b_{3,0}$

$Sq^1 b_{9,3} = b_{5,3}^2 + b_{5,2}^2$

$Sq^2 b_{9,3} = b_{3,1}^2 b_{5,1} + b_{3,0}^2 b_{5,3} + b_{6,5}b_{5,0} + b_{6,4}b_{5,2}$

$Sq^4 b_{9,3} = b_{3,1}b_{5,2}^2 + b_{3,1}b_{5,1}^2 + b_{3,0}b_{5,3}^2 + b_{3,0}b_{5,2}^2 + b_{3,0}b_{5,0}^2$

$Sq^8 b_{9,3} = b_{3,1}^4 b_{5,1} + b_{3,0}b_{5,2}b_{9,2} + b_{3,0}^3 b_{3,1}b_{5,3} + b_{3,0}^4 b_{5,3} + b_{6,5}^2 b_{5,0} + b_{6,4}b_{3,0}b_{3,1}b_{5,2} + b_{6,4}^2 b_{5,3} + c_{12,14}b_{5,3} + c_{12,14}b_{5,2} + c_{12,10}b_{5,2} + c_{8,6}b_{3,0}^3 + b_{6,5}c_{8,6}b_{3,0}$

$Sq^1 c_{12,10} = b_{3,1}b_{5,3}^2 + b_{3,1}b_{5,2}^2 + b_{3,1}b_{5,1}^2 + b_{3,1}b_{5,0}^2 + b_{3,0}b_{5,0}^2$

$Sq^2 c_{12,10} = b_{5,2}b_{9,2} + b_{5,0}b_{9,0} + b_{3,1}^3 b_{5,0} + b_{3,0}^2 b_{3,1}b_{5,3} + b_{3,0}^2 b_{3,1}b_{5,2} + b_{3,0}^3 b_{5,2} + b_{6,5}b_{3,0}b_{5,0} + b_{6,4}b_{3,1}b_{5,3} + b_{6,4}b_{3,0}b_{5,2} + b_{6,0}b_{3,1}b_{5,1} + b_{6,0}b_{3,1}b_{5,0} + c_{8,6}b_{3,0}^2 + b_{6,5}c_{8,6} + b_{6,4}c_{8,6} + a_{2,1}c_{12,14} + a_{2,0}c_{12,14}$

$Sq^4 c_{12,10} = b_{3,1}^2 b_{5,1}^2 + b_{3,1}^2 b_{5,0}b_{5,1} + b_{3,1}^2 b_{5,0}^2 + b_{3,0}b_{3,1}b_{5,2}^2 + b_{3,0}^2 b_{5,2}b_{5,3} + b_{6,5}b_{5,0}b_{5,1} + b_{6,5}b_{5,0}^2 + b_{6,4}b_{5,2}b_{5,3} + b_{6,4}b_{5,2}^2 + b_{6,0}b_{5,1}^2 + b_{6,0}b_{5,0}^2$

$Sq^8 c_{12,10} = b_{5,2}^3 b_{5,3} + b_{5,2}^4 + b_{5,0}^4 + b_{3,1}^5 b_{5,0} + b_{3,0}^4 b_{3,1}b_{5,3} + b_{3,0}^4 b_{3,1}b_{5,2} + b_{3,0}^5 b_{5,3} + b_{6,5}^2 b_{3,0}b_{5,0} + b_{6,4}b_{3,0}^2 b_{3,1}b_{5,2} + b_{6,4}^2 b_{3,0}b_{5,3} + b_{6,0}b_{5,0}b_{9,1} + b_{6,0}b_{5,0}b_{9,0} + b_{6,0}b_{3,1}^3 b_{5,1} + b_{6,0}b_{3,1}^3 b_{5,0} + b_{6,0}^2 b_{3,1}b_{5,1} + b_{6,0}^2 b_{3,1}b_{5,0} + c_{12,14}b_{3,1}b_{5,3} + c_{12,14}b_{3,1}b_{5,2} + c_{12,14}b_{3,1}b_{5,1} + c_{12,14}b_{3,1}b_{5,0} + c_{12,14}b_{3,0}b_{5,3} + c_{12,14}b_{3,0}b_{5,2} + c_{12,14}b_{3,0}b_{5,0} + c_{12,10}b_{3,1}b_{5,2} + c_{12,10}b_{3,1}b_{5,0} + c_{12,10}b_{3,0}b_{5,3} + c_{12,10}b_{3,0}b_{5,2} + c_{8,6}b_{3,1}b_{9,3} + c_{8,6}b_{3,1}b_{9,1} + c_{8,6}b_{3,1}b_{9,0} + c_{8,6}b_{3,1}^4 + c_{8,6}b_{3,0}^4 + c_{8,6}b_{3,0}b_{9,3} + c_{8,6}b_{3,0}b_{9,0} + b_{6,4}c_{8,6}b_{3,0}b_{3,1} + b_{6,4}c_{8,6}b_{3,0}^2 + b_{6,0}b_{6,5}c_{8,6} + b_{6,0}^2 c_{8,6} + c_{8,6}c_{12,10}$

$Sq^1 c_{12,14} = b_{3,1}b_{5,2}^2 + b_{3,1}b_{5,0}^2 + b_{3,0}b_{5,1}^2$

$Sq^2 c_{12,14} = b_{5,2}b_{9,3} + b_{5,0}b_{9,1} + b_{3,1}^3 b_{5,1} + b_{3,1}^3 b_{5,0} + b_{3,0}^3 b_{5,3} + b_{3,0}^3 b_{5,2} + b_{6,5}b_{3,0}b_{5,1} + b_{6,5}b_{3,0}b_{5,0} + b_{6,4}b_{3,1}b_{5,3} + b_{6,4}b_{3,1}b_{5,2} + b_{6,4}b_{3,0}b_{5,3} + b_{6,0}b_{3,1}b_{5,1} + b_{6,0}b_{3,0}b_{5,1} + c_{8,6}b_{3,0}^2 + b_{6,5}c_{8,6} + a_{2,1}c_{12,14} + a_{2,1}c_{12,10} + a_{2,0}c_{12,14} + a_{2,0}c_{12,10}$

$Sq^4c_{12,14} = b_{3,1}^2b_{5,0}b_{5,1} + b_{3,0}b_{3,1}b_{5,2}^2 + b_{3,0}^2b_{5,3}^2 + b_{3,0}^2b_{5,2}b_{5,3} + b_{6,5}b_{5,1}^2 + b_{6,5}b_{5,0}^2 + b_{6,4}b_{5,2}b_{5,3} + b_{6,4}b_{5,2}^2 + b_{6,0}b_{5,0}^2 + c_{8,6}^2$

$Sq^8c_{12,14} = b_{5,0}^3b_{5,1} + b_{5,0}^4 + b_{3,1}^5b_{5,0} + b_{3,0}^5b_{5,2} + b_{6,5}^2b_{3,0}b_{5,1} + b_{6,4}b_{5,2}b_{9,3} + b_{6,4}b_{5,2}b_{9,2} + b_{6,4}b_{3,0}^2b_{3,1}b_{5,3} + b_{6,4}b_{3,0}^2b_{3,1}b_{5,2} + b_{6,4}b_{3,0}^3b_{5,3} + b_{6,4}b_{3,0}^3b_{5,2} + b_{6,4}^2b_{3,1}b_{5,2} + b_{6,4}^2b_{3,0}b_{5,3} + b_{6,4}^2b_{3,0}b_{5,2} + b_{6,0}b_{5,0}b_{9,1} + b_{6,0}b_{6,5}b_{3,0}b_{5,0} + b_{6,0}^2b_{3,1}b_{5,1} + b_{6,0}^2b_{3,1}b_{5,0} + b_{6,0}^2b_{3,0}b_{5,0} + c_{12,14}b_{3,1}b_{5,3} + c_{12,14}b_{3,1}b_{5,2} + c_{12,14}b_{3,1}b_{5,1} + c_{12,14}b_{3,1}b_{5,0} + c_{12,14}b_{3,0}b_{5,3} + c_{12,14}b_{3,0}b_{5,2} + c_{12,14}b_{3,0}b_{5,0} + c_{12,10}b_{3,1}b_{5,3} + c_{12,10}b_{3,1}b_{5,2} + c_{12,10}b_{3,1}b_{5,1} + c_{12,10}b_{3,1}b_{5,0} + c_{12,10}b_{3,0}b_{5,1} + c_{8,6}b_{3,0}b_{9,2} + c_{8,6}b_{3,0}b_{9,1} + b_{6,5}^2c_{8,6} + b_{6,4}c_{8,6}b_{3,0}b_{3,1} + b_{6,0}b_{6,5}c_{8,6} + c_{8,6}c_{12,14} + c_{8,6}c_{12,10}$

6. M_{22}:

$Syl_2(M_{22}) \cong SmallGroup(128, 931)$.

Minimal generating set:

$$a_{2,0}, b_{3,0}, b_{5,0}, b_{5,1}, b_{6,0}, b_{6,2}, b_{7,0}, c_{8,2}, b_{8,3}, b_{9,1}, b_{9,3}, b_{10,0}, b_{11,6}, b_{12,6}, b_{12,7}, b_{15,1}$$

$Sq^1a_{2,0} = 0$

$Sq^1b_{3,0} = 0$

$Sq^2b_{3,0} = b_{5,1}$

$Sq^1b_{5,0} = b_{6,2}$

$Sq^2b_{5,0} = 0$

$Sq^4b_{5,0} = b_{9,1}$

$Sq^1b_{5,1} = b_{3,0}^2$

$Sq^2b_{5,1} = 0$

$Sq^4b_{5,1} = b_{9,3} + b_{9,1} + b_{3,0}^3 + b_{6,2}b_{3,0}$

$Sq^1b_{6,0} = b_{7,0} + a_{2,0}b_{5,0}$

$Sq^2b_{6,0} = b_{3,0}b_{5,1} + b_{3,0}b_{5,0} + b_{8,3}$

$$Sq^4 b_{6,0} = b_{5,0}^2 + b_{10,0} + a_{2,0}c_{8,2}$$

$$Sq^1 b_{6,2} = 0$$

$$Sq^2 b_{6,2} = 0$$

$$Sq^4 b_{6,2} = b_{5,0}^2$$

$$Sq^1 b_{7,0} = 0$$

$$Sq^2 b_{7,0} = 0$$

$$Sq^4 b_{7,0} = b_{11,6}$$

$$Sq^1 c_{8,2} = b_{3,0}^3 + b_{6,2}b_{3,0}$$

$$Sq^2 c_{8,2} = b_{5,1}^2 + b_{5,0}^2 + a_{2,0}c_{8,2}$$

$$Sq^4 c_{8,2} = b_{3,0}^4 + b_{12,6} + b_{6,2}^2 + b_{6,0}^2$$

$$Sq^1 b_{8,3} = b_{3,0}^3 + b_{6,0}b_{3,0}$$

$$Sq^2 b_{8,3} = b_{5,1}^2 + b_{5,0}^2 + a_{2,0}c_{8,2}$$

$$Sq^4 b_{8,3} = b_{3,0}b_{9,3} + b_{12,7}$$

$$Sq^1 b_{9,1} = b_{5,0}^2$$

$$Sq^2 b_{9,1} = 0$$

$$Sq^4 b_{9,1} = 0$$

$$Sq^8 b_{9,1} = b_{12,7}b_{5,0} + b_{12,6}b_{5,0} + b_{8,3}b_{9,1} + b_{6,0}b_{8,3}b_{3,0} + b_{6,0}^2 b_{5,0} + a_{2,0}b_{15,1} + c_{8,2}b_{9,1}$$

$$Sq^1 b_{9,3} = b_{5,1}^2 + b_{5,0}^2$$

$$Sq^2 b_{9,3} = b_{3,0}^2 b_{5,1} + b_{6,2}b_{5,0} + a_{2,0}b_{9,1}$$

$$Sq^4 b_{9,3} = b_{8,3}b_{5,1} + b_{8,3}b_{5,0}$$

$$Sq^8 b_{9,3} = b_{12,6}b_{5,1} + b_{12,6}b_{5,0} + b_{8,3}b_{9,3} + c_{8,2}b_{9,3}$$

$$Sq^1 b_{10,0} = b_{11,6} + b_{3,0}^2 b_{5,1} + b_{8,3}b_{3,0} + b_{6,0}b_{5,0} + a_{2,0}b_{9,1}$$

$$Sq^2 b_{10,0} = b_{6,2}^2 + b_{6,0}^2$$

$$Sq^4 b_{10,0} = b_{7,0}^2 + b_{5,0}b_{9,1} + b_{6,2}b_{3,0}b_{5,0} + b_{6,0}b_{8,3} + a_{2,0}b_{12,6} + b_{6,2}c_{8,2}$$

$Sq^8 b_{10,0} = b_{7,0}b_{11,6} + b_{10,0}b_{3,0}b_{5,0} + b_{6,2}b_{3,0}b_{9,1} + b_{6,2}b_{12,7} + b_{6,2}b_{12,6} + b_{6,2}^3 + b_{6,0}b_{3,0}b_{9,1} + b_{6,0}b_{12,7} + b_{6,0}b_{6,2}^2 + b_{6,0}^2 b_{6,2} + b_{6,0}^3 + c_{8,2}b_{10,0}$

$Sq^1 b_{11,6} = 0$

$Sq^2 b_{11,6} = b_{6,0}b_{7,0}$

$Sq^4 b_{11,6} = 0$

$Sq^8 b_{11,6} = b_{6,0}^2 b_{7,0} + c_{8,2}b_{11,6}$

$Sq^1 b_{12,6} = b_{8,3}b_{5,1} + b_{8,3}b_{5,0}$

$Sq^2 b_{12,6} = b_{5,0}b_{9,3} + b_{5,0}b_{9,1} + b_{6,2}b_{3,0}b_{5,0} + c_{8,2}b_{3,0}^2$

$Sq^4 b_{12,6} = b_{8,3}^2$

$Sq^8 b_{12,6} = b_{6,2}^2 b_{3,0}b_{5,0} + b_{6,2}^2 b_{8,3} + b_{6,0}b_{6,2}b_{8,3} + b_{6,0}^2 b_{3,0}b_{5,0} + c_{8,2}b_{3,0}^4 + c_{8,2}b_{12,6} + b_{6,2}^2 c_{8,2}$

$Sq^1 b_{12,7} = b_{10,0}b_{3,0} + b_{8,3}b_{5,0}$

$Sq^2 b_{12,7} = b_{5,0}b_{9,1} + b_{6,2}b_{3,0}b_{5,0} + b_{6,0}b_{8,3} + b_{6,2}c_{8,2}$

$Sq^4 b_{12,7} = b_{8,3}b_{3,0}b_{5,0} + b_{6,2}b_{5,0}^2 + b_{6,2}b_{10,0} + b_{6,0}b_{5,0}^2$

$Sq^8 b_{12,7} = b_{12,6}b_{3,0}b_{5,1} + b_{12,6}b_{3,0}b_{5,0} + b_{8,3}b_{12,6} + b_{6,0}^2 b_{8,3} + c_{8,2}b_{12,7}$

$Sq^1 b_{15,1} = b_{8,3}b_{3,0}b_{5,0} + b_{6,2}b_{10,0} + b_{6,0}b_{5,0}^2$

$Sq^2 b_{15,1} = b_{12,7}b_{5,0} + b_{12,6}b_{5,0} + b_{8,3}b_{9,3} + b_{8,3}b_{9,1} + b_{8,3}b_{3,0}^3 + b_{6,2}b_{8,3}b_{3,0} + b_{6,0}b_{8,3}b_{3,0} + a_{2,0}b_{15,1} + c_{8,2}b_{9,3} + c_{8,2}b_{9,1} + c_{8,2}b_{3,0}^3 + b_{6,2}c_{8,2}b_{3,0}$

$Sq^4 b_{15,1} = b_{8,3}^2 b_{3,0} + b_{6,0}^2 b_{7,0} + a_{2,0}b_{12,6}b_{5,0} + c_{8,2}b_{11,6} + c_{8,2}b_{3,0}^2 b_{5,1} + b_{6,2}c_{8,2}b_{5,0} + a_{2,0}c_{8,2}b_{9,1} + c_{8,2}^2 b_{3,0}$

$Sq^8 b_{15,1} = b_{8,3}b_{15,1} + b_{8,3}b_{5,1}^3 + b_{8,3}b_{12,6}b_{3,0} + b_{6,2}b_{8,3}b_{9,1} + b_{6,2}^2 b_{8,3}b_{3,0} + b_{6,0}b_{12,7}b_{5,0} + b_{6,0}^2 b_{11,6} + b_{6,0}^2 b_{8,3}b_{3,0} + a_{2,0}b_{12,6}b_{9,1} + c_{8,2}b_{15,1} + c_{8,2}b_{5,1}^3 + c_{8,2}b_{3,0}^5 + c_{8,2}b_{12,6}b_{3,0} + c_{8,2}b_{10,0}b_{5,0} + b_{6,0}c_{8,2}b_{9,1} + b_{6,0}b_{6,2}c_{8,2}b_{3,0}$

5.2.4 Janko groups

1. J_1:

   $Syl_2(J_1) \cong SmallGroup(8,5)$.

   Minimal generating set: $c_{3,0}, c_{4,0}, c_{5,0}, c_{6,1}, c_{7,1}$.

   $Sq^1 c_{3,0} = c_{4,0}$

   $Sq^2 c_{3,0} = c_{5,0}$

   $Sq^1 c_{4,0} = 0$

   $Sq^2 c_{4,0} = c_{6,1}$

   $Sq^1 c_{5,0} = c_{3,0}^2$

   $Sq^2 c_{5,0} = c_{7,1}$

   $Sq^4 c_{5,0} = c_{6,1}c_{3,0} + c_{4,0}c_{5,0}$

   $Sq^1 c_{6,1} = c_{7,1}$

   $Sq^2 c_{6,1} = 0$

   $Sq^4 c_{6,1} = c_{4,0}c_{6,1}$

   $Sq^1 c_{7,1} = 0$

   $Sq^2 c_{7,1} = 0$

   $Sq^4 c_{7,1} = c_{4,0}c_{7,1}$

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